

QUADRATIC RATIONAL MAPS WITH INTEGER MULTIPLIERS

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ABSTRACT. In this article, we prove that every quadratic rational map whose multipliers all lie in the ring of integers of a given imaginary quadratic field is a power map, a Chebyshev map or a Lattès map. In particular, this provides some evidence in support of a conjecture by Milnor concerning rational maps whose multipliers are all integers.

1. INTRODUCTION

Given a rational map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ and a point $z_0 \in \widehat{\mathbb{C}}$, we study the sequence $(f^{\circ n}(z_0))_{n \geq 0}$ of iterates of f at z_0 . The set $\{f^{\circ n}(z_0) : n \geq 0\}$ is called the *forward orbit* of z_0 under f .

The point z_0 is said to be *periodic* for f if there exists an integer $n \geq 1$ such that $f^{\circ n}(z_0) = z_0$; the least such integer n is called the *period* of z_0 . The forward orbit of z_0 , which has cardinality n , is said to be a *cycle* for f . The *multiplier* of f at z_0 is the unique eigenvalue of the differential of $f^{\circ n}$ at z_0 . The map f has the same multiplier at each point of the cycle.

The multiplier is invariant under conjugacy: if f and g are rational maps, ϕ is a Möbius transformation such that $\phi \circ f = g \circ \phi$ and z_0 is a periodic point for f , then $\phi(z_0)$ is a periodic point for g with the same period and the same multiplier.

We wish to examine here the rational maps that have only integer multipliers.

Definition 1. A rational map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ of degree $d \geq 2$ is said to be a *power map* if it is conjugate to $z \mapsto z^{\pm d}$.

For every $d \geq 2$, there exists a unique polynomial $T_d \in \mathbb{C}[z]$ such that

$$T_d(z + z^{-1}) = z^d + z^{-d}.$$

The polynomial T_d is monic of degree d and is called the *dth Chebyshev polynomial*.

Definition 2. A rational map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ of degree $d \geq 2$ is said to be a *Chebyshev map* if it is conjugate to $\pm T_d$.

Remark 3. For every $d \geq 2$, the rational maps $-T_d$ and T_d are conjugate if and only if d is even.

These rational maps share the following well-known property:

Proposition 4. *Suppose that $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a power map or a Chebyshev map. Then f has only integer multipliers.*

In fact, there exist also other rational maps that satisfy this special condition.

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Definition 5. A rational map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ of degree $d \geq 2$ is said to be a *Lattès map* if there exist a torus $\mathbb{T} = \mathbb{C}/\Lambda$, with Λ a lattice in \mathbb{C} , a holomorphic map $L: \mathbb{T} \rightarrow \mathbb{T}$ and a nonconstant holomorphic map $p: \mathbb{T} \rightarrow \widehat{\mathbb{C}}$ that make the following diagram commute:

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{L} & \mathbb{T} \\ p \downarrow & & \downarrow p \\ \widehat{\mathbb{C}} & \xrightarrow{f} & \widehat{\mathbb{C}} \end{array}$$

Remark 6. Suppose that Λ is a lattice in \mathbb{C} and $\mathbb{T} = \mathbb{C}/\Lambda$. Then the holomorphic maps $L: \mathbb{T} \rightarrow \mathbb{T}$ are precisely the maps of the form

$$L_{a,b}^\Lambda: z + \Lambda \mapsto az + b + \Lambda, \quad \text{with } a, b \in \mathbb{C}, a\Lambda \subset \Lambda.$$

Moreover, for all $a, b \in \mathbb{C}$ such that $a\Lambda \subset \Lambda$, the map $L_{a,b}^\Lambda: \mathbb{T} \rightarrow \mathbb{T}$ has degree $|a|^2$.

We distinguish two types of Lattès maps. A rational map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ of degree $d \geq 2$ is said to be a *flexible* Lattès map if there exist a torus $\mathbb{T} = \mathbb{C}/\Lambda$, with Λ a lattice in \mathbb{C} , $a \in \mathbb{Z}$, $b \in \mathbb{C}$ and a holomorphic map $p: \mathbb{T} \rightarrow \widehat{\mathbb{C}}$ of degree 2 such that

$$f \circ p = p \circ L_{a,b}^\Lambda, \quad \text{where } L_{a,b}^\Lambda: z + \Lambda \mapsto az + b + \Lambda.$$

A non-flexible Lattès map is said to be *rigid*. We refer the reader to [Mil06] or [Sil07, Chapter 6] for further information about Lattès maps.

Remark 7. The degree of a flexible Lattès map is the square of an integer.

Given a positive squarefree integer D , we denote by R_D the ring of integers of the imaginary quadratic field $\mathbb{Q}(i\sqrt{D})$.

Lattès maps have the following remarkable property:

Proposition 8 ([Mil06, Corollary 3.9 and Lemma 5.6]). *Suppose that $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a Lattès map. Then there exists a positive squarefree integer D such that the multipliers of f all lie in R_D . Furthermore, the multipliers of f are all integers if and only if f is flexible.*

We are interested in the converse of Proposition 4 and Proposition 8. In [Mil06], Milnor conjectured that power maps, Chebyshev maps and flexible Lattès maps are the only rational maps whose multipliers are all integers. More generally, we may wonder whether power maps, Chebyshev maps and Lattès maps are the only rational maps whose multipliers all lie in the ring of integers of a given imaginary quadratic field. We answer this question in the case of quadratic rational maps.

Theorem 9. *Assume that D is a positive squarefree integer and $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a quadratic rational map whose multiplier at each cycle with period less than or equal to 5 lies in R_D . Then f is a power map, a Chebyshev map or a Lattès map.*

In particular, this proves Milnor's conjecture for quadratic rational maps.

Corollary 10. *Assume that $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a quadratic rational map that has only integer multipliers. Then f is either a power map or a Chebyshev map.*

We may even extend Milnor’s question as follows:

Question 11. Assume that K is a number field, \mathcal{O}_K is its ring of integers and $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a rational map of degree $d \geq 2$ whose multipliers all lie in \mathcal{O}_K – or K . Is f necessarily a power map, a Chebyshev map or a Lattès map?

In [Hug21b], the author answered this question for certain polynomial maps. More precisely, he proved that every unicritical polynomial map of degree $d \geq 2$ that has only rational multipliers is either a power map or a Chebyshev map. He also proved that every cubic polynomial map with symmetries that has only integer multipliers is either a power map or a Chebyshev map.

In [EvS11], Eremenko and van Strien studied the rational maps of degree $d \geq 2$ that have only real multipliers: they proved that, if $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is such a map, then either f is a Lattès map or its Julia set \mathcal{J}_f is contained in a circle; they also gave a description of these maps.

In Section 2, we provide some background about the multiplier polynomials of a rational map, the moduli space of quadratic rational maps and the ring of integers of an imaginary quadratic field.

In Section 3, we prove Theorem 9. More precisely, we determine the quadratic rational maps whose multiplier polynomials all split into linear factors in $R_D[\lambda]$, with D a given positive squarefree integer. Using the holomorphic fixed-point formula, we are reduced to studying two one-parameter families of rational maps and finitely many other cases. We then examine the multiplier polynomials associated to these two families and to the remaining cases in order to conclude.

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2. SOME PRELIMINARIES

We shall review here some necessary material for our proof of Theorem 9.

2.1. Dynatomic polynomials and multiplier polynomials. First, we present the dynatomic polynomials and the multiplier polynomials associated to a rational map, which are related to its periodic points and its multipliers. In particular, we provide a formula to compute the multiplier polynomials of a rational map, which will be very useful in our proof of Theorem 9. For further information about these polynomials, we refer the reader to [MS95], [Sil98] and [Sil07, Chapter 4].

Throughout this subsection, we fix an integer $d \geq 2$, which will denote the degree of a rational map. In order to properly take the point ∞ into account, we identify the Riemann sphere $\widehat{\mathbb{C}}$ with the complex projective line $\mathbb{P}^1(\mathbb{C})$ – defined as the quotient of $\mathbb{C}^2 \setminus \{0\}$ by the relation of collinearity – by the usual biholomorphism $\iota: \widehat{\mathbb{C}} \rightarrow \mathbb{P}^1(\mathbb{C})$ and its inverse given by

$$\iota(z) = \begin{cases} [z: 1] & \text{if } z \in \mathbb{C} \\ [1: 0] & \text{if } z = \infty \end{cases} \quad \text{and} \quad \iota^{-1}([x: y]) = \begin{cases} \frac{x}{y} & \text{if } y \in \mathbb{C}^* \\ \infty & \text{if } y = 0 \end{cases} .$$

Suppose that $f: \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$ is a rational map of degree d . Then there exists a homogeneous polynomial map $F: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ that does not vanish on $\mathbb{C}^2 \setminus \{0\}$ and makes the diagram below commute, where π denotes the canonical projection. The map F is unique up to multiplication by an element of \mathbb{C}^* and is said to be a *homogeneous polynomial lift* of f .

$$\begin{array}{ccc}
\mathbb{C}^2 \setminus \{0\} & \xrightarrow{F} & \mathbb{C}^2 \setminus \{0\} \\
\pi \downarrow & & \downarrow \pi \\
\mathbb{P}^1(\mathbb{C}) & \xrightarrow{f} & \mathbb{P}^1(\mathbb{C})
\end{array}$$

Given a homogeneous polynomial map $F: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ of degree d and $n \geq 0$, we denote by G_n^F and H_n^F the polynomials in $\mathbb{C}[x, y]$ defined by

$$F^{\circ n}(x, y) = (G_n^F(x, y), H_n^F(x, y)) ,$$

which are homogeneous of degree d^n .

Suppose that $f: \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$ is a rational map of degree d and $F: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is a homogeneous polynomial lift of f . Then, for every $n \geq 1$, the roots in $\mathbb{P}^1(\mathbb{C})$ of the homogeneous polynomial

$$yG_n^F(x, y) - xH_n^F(x, y) \in \mathbb{C}[x, y]$$

are precisely the periodic points for f with period dividing n . Thus, it is natural to try to factor these polynomials in order to separate their roots according to their periods, and we obtain the result below.

For $n \geq 1$, we define

$$\nu(n) = \sum_{k|n} \mu\left(\frac{n}{k}\right) d^k ,$$

where $\mu: \mathbb{Z}_{\geq 1} \rightarrow \{-1, 0, 1\}$ denotes the Möbius function.

Proposition 12 ([MS95, Proposition 3.2]). *Suppose that $F: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is a homogeneous polynomial map of degree d that does not vanish on $\mathbb{C}^2 \setminus \{0\}$. Then there exists a unique sequence $(\Phi_n^F)_{n \geq 1}$ of elements of $\mathbb{C}[x, y]$ such that, for every $n \geq 1$, we have*

$$yG_n^F(x, y) - xH_n^F(x, y) = \prod_{k|n} \Phi_k^F(x, y) .$$

Furthermore, for every $n \geq 1$, the polynomial Φ_n^F is nonzero and homogeneous and we have

$$\deg \Phi_n^F = \begin{cases} d+1 & \text{if } n=1 \\ \nu(n) & \text{if } n \geq 2 \end{cases} .$$

Definition 13. Suppose that $F: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is a homogeneous polynomial map of degree d that does not vanish on $\mathbb{C}^2 \setminus \{0\}$. For $n \geq 1$, the polynomial Φ_n^F is called the *n*th *dynamotic polynomial* of F .

Remark 14. If $F: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is a homogeneous polynomial map of degree d that does not vanish on $\mathbb{C}^2 \setminus \{0\}$, then we have

$$\Phi_n^F(x, y) = \prod_{k|n} (yG_k^F(x, y) - xH_k^F(x, y))^{\mu\left(\frac{n}{k}\right)}$$

for all $n \geq 1$ by the Möbius inversion formula.

The following result gives the relation between the periodic points for a rational map and the dynamotic polynomials of its homogeneous polynomial lifts.

Proposition 15 ([MS95, Proposition 3.2]). *Assume that $f: \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$ is a rational map of degree d , $F: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is a homogeneous polynomial lift of f and $n \geq 1$. Then $z_0 \in \mathbb{P}^1(\mathbb{C})$ is a root of the polynomial Φ_n^F if and only if z_0 is either a periodic point for f with period n or a periodic point for f with period a proper divisor k of n and multiplier a primitive $\frac{n}{k}$ th root of unity.*

Let us now present the multiplier polynomials of a rational map. Suppose that $f: \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$ is a rational map of degree d and $n \geq 1$. Informally, we want to compute the polynomial

$$\prod_{j=1}^r (\lambda - \lambda_j) \in \mathbb{C}[\lambda],$$

where $\lambda_1, \dots, \lambda_r$ denote the multipliers of f at its periodic points with period n . In fact, since f has the same multiplier at each point of a cycle, we want to obtain the n th root of this polynomial. Assume that $z_0 \in \mathbb{P}^1(\mathbb{C})$ is a periodic point for f with period n and multiplier λ_0 and $F: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is a homogeneous polynomial lift of f . Then there exists a periodic point $(x_0, y_0) \in z_0$ for F with period n , and the eigenvalues of the differential of $F^{\circ n}$ at (x_0, y_0) are precisely d^n and λ_0 . Therefore, considering the trace of the differential of $F^{\circ n}$ at (x_0, y_0) , we have

$$\lambda_0 + d^n = T_n^F(x_0, y_0),$$

where

$$T_n^F = \frac{\partial G_n^F}{\partial x} + \frac{\partial H_n^F}{\partial y} \in \mathbb{C}[x, y].$$

This discussion leads us to the result below (see [Hug21a, Chapitre 3]).

Proposition 16. *Suppose that $f: \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$ is a rational map of degree d , $F: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is a homogeneous polynomial lift of f and $n \geq 1$. Then there exists a unique monic polynomial $M_n^f \in \mathbb{C}[\lambda]$ such that, for every homogeneous polynomial $P \in \mathbb{C}[x, y]$ of degree 1, we have*

$$\text{res}(\Phi_n^F, P \circ F^{\circ n}) M_n^f(\lambda)^n = \text{res}(\Phi_n^F, (\lambda + d^n)P \circ F^{\circ n} - PT_n^F),$$

where res denotes the homogeneous resultant. Furthermore, M_n^f depends only on f and we have

$$\deg M_n^f = \begin{cases} d+1 & \text{if } n=1 \\ \frac{\nu(n)}{n} & \text{if } n \geq 2 \end{cases}.$$

Definition 17. Suppose that $f: \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$ is a rational map of degree d . For $n \geq 1$, the polynomial M_n^f is called the n th *multiplier polynomial* of f .

Remark 18. If $F: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is a homogeneous polynomial map of degree d that does not vanish on $\mathbb{C}^2 \setminus \{0\}$, $n \geq 1$ and $P \in \mathbb{C}[x, y]$ is a homogeneous polynomial of degree $e \geq 0$, then we have

$$\text{res}(\Phi_n^F, P \circ F^{\circ n}) = \epsilon_n^{(e)} \text{res}(\Phi_n^F, P) \text{res}(F)^{\frac{e\nu(n)(d^n-1)}{d(d-1)}},$$

where $\epsilon_n^{(e)} \in \{-1, 1\}$ equals -1 if and only if $n=1$, d is even and e is odd.

Note that, given a rational map $f: \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$ of degree d , a homogeneous polynomial lift $F: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ of f and $n \geq 1$, the formula in Proposition 16 enables us to compute the polynomial M_n^f by considering a nonzero homogeneous polynomial $P \in \mathbb{C}[x, y]$ of degree 1 that does not divide Φ_n^F .

Let us now describe precisely the relation between the multiplier polynomials of a rational map and its multipliers. Given a rational map $f: \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$ of degree d , a homogeneous polynomial lift $F: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ of f and $n \geq 1$, we have

$$M_n^f(\lambda)^n = \prod_{j=1}^{\deg \Phi_n^F} (\lambda - \lambda_j),$$

where $\lambda_1, \dots, \lambda_{\deg \Phi_n^F}$ are the multipliers of $f^{\circ n}$ at the roots of the polynomial Φ_n^F repeated according to their multiplicities. Therefore, we have the result below, which follows immediately from Proposition 15.

Proposition 19. *Assume that $f: \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$ is a rational map of degree d and $n \geq 1$. Then $\lambda_0 \in \mathbb{C}$ is a root of the polynomial M_n^f if and only if*

- λ_0 is the multiplier of f at a cycle with period n ,
- or λ_0 equals 1 and f has a cycle with period a proper divisor k of n and multiplier a primitive $\frac{n}{k}$ th root of unity.

A direct consequence of Proposition 19 is the result below, which is a key point in our proof of Theorem 9. It states that our problem comes down to determining the quadratic rational maps whose multiplier polynomials all split into linear factors in $R_D[\lambda]$, with D a given positive squarefree integer.

Corollary 20. *Assume that R is a subring of \mathbb{C} , $f: \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$ is a rational map of degree d and $n \geq 1$. Then the multipliers of f at its cycles with period n all lie in R if and only if the polynomial M_n^f splits into linear factors in $R[\lambda]$.*

2.2. The moduli space of quadratic rational maps. We now recall certain facts about the conjugacy classes of quadratic rational maps.

Suppose that $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a quadratic rational map, and denote by $\lambda_1, \lambda_2, \lambda_3$ its multipliers at its fixed points repeated according to their multiplicities. If f has only simple fixed points or, equivalently, if $\lambda_j \neq 1$ for all $j \in \{1, 2, 3\}$, then we have

$$\frac{1}{1 - \lambda_1} + \frac{1}{1 - \lambda_2} + \frac{1}{1 - \lambda_3} = 1.$$

In particular, note that $\lambda_1 \lambda_2 = 1$ if and only if $\lambda_1 = \lambda_2 = 1$ since it follows that

$$(1 - \lambda_1)(1 - \lambda_2) = (\lambda_1 \lambda_2 - 1)(1 - \lambda_3),$$

which holds even if f has a multiple fixed point.

Given a quadratic rational map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$, we define

$$\sigma_1^f = \lambda_1 + \lambda_2 + \lambda_3, \quad \sigma_2^f = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3, \quad \sigma_3^f = \lambda_1 \lambda_2 \lambda_3$$

to be the elementary symmetric functions of the multipliers $\lambda_1, \lambda_2, \lambda_3$ of f at its fixed points, so that

$$M_1^f(\lambda) = \lambda^3 - \sigma_1^f \lambda^2 + \sigma_2^f \lambda - \sigma_3^f.$$

By the formula above that relates the multipliers of a quadratic rational map at its fixed points, for every quadratic rational map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$, we have

$$\sigma_3^f = \sigma_1^f - 2.$$

In fact, we will see that this relation uniquely determines the conjugacy classes of quadratic rational maps.

We now give normal forms for the conjugacy classes of quadratic rational maps. For $a, b \in \mathbb{C}$ such that $ab \neq 1$, define

$$g_{a,b}: z \mapsto \frac{z(z+a)}{bz+1},$$

which fixes 0 with multiplier a and fixes ∞ with multiplier b . Define

$$h: z \mapsto z + \frac{1}{z},$$

which has ∞ as its unique fixed point. We have the following result:

Proposition 21 ([Mil93, Lemma 3.1]). *Suppose that $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a quadratic rational map. If f has two distinct fixed points with multipliers $a, b \in \mathbb{C}$, then we have $ab \neq 1$ and f is conjugate to $g_{a,b}$. If f has a unique fixed point, then f is conjugate to h .*

We will also use another normal form. For $c \in \mathbb{C}$, define

$$f_c: z \mapsto z^2 + c.$$

For every $c \in \mathbb{C}$, the map f_c has ∞ as a superattracting fixed point. Furthermore, if $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a quadratic rational map that has a superattracting fixed point, then there exists a unique parameter $c \in \mathbb{C}$ such that f is conjugate to f_c . Note that, for every $c \in \mathbb{C}$, the map f_c is a power map if and only if $c = 0$ and is a Chebyshev map if and only if $c = -2$.

Define $\mathcal{M}_2(\mathbb{C})$ to be the set of conjugacy classes of quadratic rational maps. Given a quadratic rational map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$, denote by $[f] \in \mathcal{M}_2(\mathbb{C})$ its conjugacy class. We have the result below, which follows directly from Proposition 21.

Corollary 22 ([Mil93, Lemma 3.1]). *The map $\text{Mult}_2^{(1)}: \mathcal{M}_2(\mathbb{C}) \rightarrow \mathbb{C}^2$ given by*

$$\text{Mult}_2^{(1)}([f]) = (\sigma_1^f, \sigma_2^f)$$

is well defined and bijective. In particular, the conjugacy class of a quadratic rational map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is characterized by its multipliers $\lambda_1, \lambda_2, \lambda_3$ at its fixed points.

By Corollary 22 and the invariance of the multiplier under conjugacy, the multiplier polynomials of f , with $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ a quadratic rational map, depend only on σ_1^f and σ_2^f . More precisely, we have the following result:

Proposition 23 ([Sil98, Corollary 5.2]). *Assume that $n \geq 1$. Then the coefficients of the polynomial M_n^f , with $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ a quadratic rational map, are polynomials in σ_1^f and σ_2^f with integer coefficients – which are independent of f .*

Remark 24. If f and \bar{f} are quadratic rational maps with multipliers $\lambda_1, \lambda_2, \lambda_3$ and $\overline{\lambda_1}, \overline{\lambda_2}, \overline{\lambda_3}$ at their fixed points, then $M_n^{\bar{f}} = \overline{M_n^f}$ for all $n \geq 1$ by Proposition 23.

Using the software SageMath, we can compute the first multiplier polynomials of $g_{a,b}$, with $a, b \in \mathbb{C}$ such that $ab \neq 1$. Thus, we can express the first multiplier polynomials of a quadratic rational map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ in terms of σ_1^f and σ_2^f .

Example 25. Suppose that $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a quadratic rational map. For simplicity, set $\sigma_1 = \sigma_1^f$ and $\sigma_2 = \sigma_2^f$, so that

$$M_1^f(\lambda) = \lambda^3 - \sigma_1 \lambda^2 + \sigma_2 \lambda - (\sigma_1 - 2).$$

For $n \geq 1$, write

$$M_n^f(\lambda) = \lambda^{\deg M_n^f} + \sum_{j=1}^{\deg M_n^f} (-1)^j \sigma_j^{(n)} \lambda^{\deg M_n^f - j}.$$

Then we have

$$\begin{aligned} \sigma_1^{(2)} &= 2\sigma_1 + \sigma_2, \\ \sigma_1^{(3)} &= \sigma_1(2\sigma_1 + \sigma_2) + 3\sigma_1 + 2, \\ \sigma_2^{(3)} &= (2\sigma_1 + \sigma_2)(\sigma_1 + \sigma_2)^2 - \sigma_1(\sigma_1 + 2\sigma_2) + 12\sigma_1 + 28, \\ \sigma_1^{(4)} &= (2\sigma_1 + \sigma_2)\sigma_1^2 + (\sigma_1 - \sigma_2)(3\sigma_1 + \sigma_2) + 10\sigma_1, \\ \sigma_2^{(4)} &= (2\sigma_1 + \sigma_2)\sigma_1^2(\sigma_1 + \sigma_2)^2 + (\sigma_1 - \sigma_2)(7\sigma_1^3 + 9\sigma_1^2\sigma_2 + 5\sigma_1\sigma_2^2 + \sigma_2^3) \\ &\quad + (26\sigma_1 - \sigma_2)\sigma_1^2 + 4\sigma_1(16\sigma_1 - 5\sigma_2) + 4(10\sigma_1 - 13\sigma_2) + 48, \\ \sigma_3^{(4)} &= \sigma_2^2(\sigma_1 + \sigma_2)^2(2\sigma_1 + \sigma_2)^2 + \sigma_1(2\sigma_1 + \sigma_2)(\sigma_1^3 - 2\sigma_1^2\sigma_2 - \sigma_1\sigma_2^2 - 2\sigma_2^3) \\ &\quad + \sigma_1(27\sigma_1^3 + 30\sigma_1^2\sigma_2 + 68\sigma_1\sigma_2^2 + 28\sigma_2^3) + 4(26\sigma_1^3 + \sigma_1^2\sigma_2 + 32\sigma_1\sigma_2^2 + 15\sigma_2^3) \\ &\quad + 8(37\sigma_1^2 - 19\sigma_1\sigma_2 - 6\sigma_2^2) + 32(20\sigma_1 + 3\sigma_2) + 304. \end{aligned}$$

Finally, let us describe the conjugacy classes of Lattès maps of degree 2. Suppose that Λ is a lattice in \mathbb{C} , and set $\mathbb{T} = \mathbb{C}/\Lambda$. Recall that the Weierstrass's function $\wp_\Lambda: \mathbb{T} \rightarrow \widehat{\mathbb{C}}$ given by

$$\wp_\Lambda(z + \Lambda) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

is well defined, even and holomorphic of degree 2. Therefore, for all $a, b \in \mathbb{C}$ such that $a\Lambda \subset \Lambda$ and $2b \in \Lambda$, there exists a unique rational map $\text{Lat}_{a,b}^{\Lambda,2}: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ of degree $|a|^2$ such that

$$\text{Lat}_{a,b}^{\Lambda,2} \circ \wp_\Lambda = \wp_\Lambda \circ L_{a,b}^\Lambda, \quad \text{where } L_{a,b}^\Lambda: z + \Lambda \mapsto az + b + \Lambda,$$

since $L_{a,b}^\Lambda$ commutes with the multiplication by -1 in \mathbb{T} .

Note that certain lattices in \mathbb{C} are invariant by nontrivial rotations about the origin, which gives rise to other Lattès maps. Suppose that $\Lambda = \mathbb{Z}[i]$ and $\mathbb{T} = \mathbb{C}/\Lambda$. Then, for every $z \in \mathbb{C}$, we have

$$\wp_\Lambda(iz + \Lambda) = -\wp_\Lambda(z + \Lambda).$$

Therefore, for all $a, b \in \mathbb{C}$ such that $a \in \Lambda$ and $(1+i)b \in \Lambda$, there exists a unique rational map $\text{Lat}_{a,b}^{\Lambda,4}: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ of degree $|a|^2$ such that

$$\text{Lat}_{a,b}^{\Lambda,4} \circ \wp_\Lambda^2 = \wp_\Lambda^2 \circ L_{a,b}^\Lambda$$

since $L_{a,b}^\Lambda$ commutes with the multiplication by i in \mathbb{T} .

We can now explicit the Lattès maps of degree 2 up to conjugacy.

Proposition 26 ([Mil06, Subsection 8.1]). *Assume that $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a Lattès map of degree 2. Then f is conjugate to either $\text{Lat}_{1+i,0}^{\mathbb{Z}[i],4}$ or $\text{Lat}_{a,b}^{\Lambda,2}$, with*

$$\Lambda \in \left\{ \mathbb{Z}[i], \mathbb{Z}[i\sqrt{2}], \mathbb{Z}\left[\frac{1+i\sqrt{7}}{2}\right] \right\}$$

TABLE 1. Multipliers $\lambda_1, \lambda_2, \lambda_3$ of $\text{Lat}_{a,b}^{\Lambda,n}$ at its fixed points.

Λ	n	a	b	$\lambda_1, \lambda_2, \lambda_3$
$\mathbb{Z}[i]$	2	$1-i$	0	$-1+i, -1+i, -2i$
$\mathbb{Z}[i]$	2	$1+i$	0	$-1-i, -1-i, 2i$
$\mathbb{Z}[i]$	4	$1+i$	0	$-4, -1-i, -1+i$
$\mathbb{Z}[i\sqrt{2}]$	2	$i\sqrt{2}$	0	$-2, -i\sqrt{2}, i\sqrt{2}$
$\mathbb{Z}\left[\frac{1+i\sqrt{7}}{2}\right]$	2	$\frac{1-i\sqrt{7}}{2}$	0	$\frac{-3-i\sqrt{7}}{2}, \frac{-3-i\sqrt{7}}{2}, \frac{-1+i\sqrt{7}}{2}$
$\mathbb{Z}\left[\frac{1+i\sqrt{7}}{2}\right]$	2	$\frac{1-i\sqrt{7}}{2}$	$\frac{1}{2}$	$\frac{-1+i\sqrt{7}}{2}, \frac{-1+i\sqrt{7}}{2}, \frac{1-i\sqrt{7}}{2}$
$\mathbb{Z}\left[\frac{1+i\sqrt{7}}{2}\right]$	2	$\frac{1+i\sqrt{7}}{2}$	0	$\frac{-3+i\sqrt{7}}{2}, \frac{-3+i\sqrt{7}}{2}, \frac{-1-i\sqrt{7}}{2}$
$\mathbb{Z}\left[\frac{1+i\sqrt{7}}{2}\right]$	2	$\frac{1+i\sqrt{7}}{2}$	$\frac{1}{2}$	$\frac{-1-i\sqrt{7}}{2}, \frac{-1-i\sqrt{7}}{2}, \frac{1+i\sqrt{7}}{2}$

and

$$(a, b) \in \begin{cases} \{(1-i, 0), (1+i, 0)\} & \text{if } \Lambda = \mathbb{Z}[i] \\ \{(i\sqrt{2}, 0)\} & \text{if } \Lambda = \mathbb{Z}[i\sqrt{2}] \\ \left\{ \left(\frac{1-i\sqrt{7}}{2}, 0\right), \left(\frac{1-i\sqrt{7}}{2}, \frac{1}{2}\right), \left(\frac{1+i\sqrt{7}}{2}, 0\right), \left(\frac{1+i\sqrt{7}}{2}, \frac{1}{2}\right) \right\} & \text{if } \Lambda = \mathbb{Z}\left[\frac{1+i\sqrt{7}}{2}\right] \end{cases}.$$

We can compute the multipliers of the Lattès maps appearing in Proposition 26 at their fixed points (see Table 1). Thus, we have the result below, which follows immediately from Corollary 22 and Proposition 26 and gives a characterization of the Lattès maps of degree 2.

Corollary 27. *Assume that $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a quadratic rational map. Then f is a Lattès map if and only if its multipliers at its fixed points are*

- either $-4, -1-i$ and $-1+i$,
- or $-1-i, -1-i$ and $2i$,
- or $-1+i, -1+i$ and $-2i$,
- or $-2, -i\sqrt{2}$ and $i\sqrt{2}$,
- or $\frac{-3-i\sqrt{7}}{2}, \frac{-3-i\sqrt{7}}{2}$ and $\frac{-1+i\sqrt{7}}{2}$,
- or $\frac{-3+i\sqrt{7}}{2}, \frac{-3+i\sqrt{7}}{2}$ and $\frac{-1-i\sqrt{7}}{2}$,
- or $\frac{-1-i\sqrt{7}}{2}, \frac{-1-i\sqrt{7}}{2}$ and $\frac{1+i\sqrt{7}}{2}$,
- or $\frac{-1+i\sqrt{7}}{2}, \frac{-1+i\sqrt{7}}{2}$ and $\frac{1-i\sqrt{7}}{2}$.

2.3. The ring of integers of an imaginary quadratic field. Finally, we recall here some properties of the ring of integers R_D of the imaginary quadratic field $\mathbb{Q}(i\sqrt{D})$, with D a positive squarefree integer.

Assume that D is a positive squarefree integer. Then we have

$$R_D = \mathbb{Z}[\alpha_D], \quad \text{where } \alpha_D = \begin{cases} i\sqrt{D} & \text{if } D \equiv 1, 2 \pmod{4} \\ \frac{1+i\sqrt{D}}{2} & \text{if } D \equiv 3 \pmod{4} \end{cases}.$$

The ring R_D is an integrally closed domain. In particular, if $a, b \in R_D$ are such that ab^2 is a square in R_D , then a is a square in R_D or b is zero. This property will be useful in our proof of Theorem 9.

The elements of R_D form a lattice in \mathbb{C} . Let us describe the intersections of R_D with the Euclidean disks centered at the origin. We denote by $N: \mathbb{C} \rightarrow \mathbb{R}_{\geq 0}$ the

map given by $N(z) = |z|^2$, which is multiplicative and agrees with the norm of the extension $\mathbb{Q}(i\sqrt{D})/\mathbb{Q}$.

Suppose that $D \equiv 1, 2 \pmod{4}$. For all $x, y \in \mathbb{Z}$, we have

$$N(x + y\alpha_D) = x^2 + Dy^2 \in \mathbb{Z}_{\geq 0}.$$

Therefore, for every $B \geq 0$, we have

$$\{z \in R_D : N(z) \leq B\} \subset \left\{ x + y\alpha_D : x, y \in \mathbb{Z}, |x| \leq \sqrt{B}, |y| \leq \sqrt{\frac{B}{D}} \right\},$$

and in particular

$$\{z \in R_D : N(z) \leq B\} \subset \mathbb{Z} \quad \text{if } B < D.$$

Suppose that $D \equiv 3 \pmod{4}$. For all $x, y \in \mathbb{Z}$, we have

$$N(x + y\alpha_D) = \left(x + \frac{1}{2}y\right)^2 + \frac{D}{4}y^2 = x^2 + xy + \frac{D+1}{4}y^2 \in \mathbb{Z}_{\geq 0}.$$

Therefore, for every $B \geq 0$, we have

$$\{z \in R_D : N(z) \leq B\} \subset \left\{ x + y\alpha_D : x, y \in \mathbb{Z}, |x| \leq \sqrt{B} + \sqrt{\frac{B}{D}}, |y| \leq 2\sqrt{\frac{B}{D}} \right\},$$

and in particular

$$\{z \in R_D : N(z) \leq B\} \subset \mathbb{Z} \quad \text{if } 4B < D.$$

Thus, the set of all imaginary quadratic integers is a discrete subset of \mathbb{C} and, for every $B \geq 0$, we can determine the pairs $(D, z) \in \mathbb{Z} \times \mathbb{C}$ such that D is a positive squarefree integer, $z \in R_D$ and $N(z) \leq B$.

3. PROOF OF THE RESULT

We shall prove here Theorem 9. It follows directly from the three lemmas below.

Lemma 28. *Assume that D is a positive squarefree integer and $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a quadratic rational map that has no superattracting or multiple fixed point and whose multiplier at each cycle with period less than or equal to 5 lies in R_D . Then f is either a power map or a Lattès map.*

Proof. Denote by $\lambda_1, \lambda_2, \lambda_3$ the multipliers of f at its fixed points, which belong to $R_D \setminus \{0, 1\}$ by hypothesis. Then $1 - \lambda_j$ lies in $R_D \setminus \{0, 1\}$ for all $j \in \{1, 2, 3\}$ and we have

$$\frac{1}{1 - \lambda_1} + \frac{1}{1 - \lambda_2} + \frac{1}{1 - \lambda_3} = 1.$$

If μ_1, μ_2, μ_3 are elements of $R_D \setminus \{0, 1\}$ that satisfy

$$\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} = 1 \quad \text{and} \quad \Re\left(\frac{1}{\mu_1}\right) \leq \Re\left(\frac{1}{\mu_2}\right) \leq \Re\left(\frac{1}{\mu_3}\right),$$

then we have

$$\Re\left(\frac{1}{\mu_3}\right) \geq \frac{1}{3}, \quad \Re\left(\frac{1}{\mu_2}\right) \geq \frac{1}{2} \left(1 - \Re\left(\frac{1}{\mu_3}\right)\right) \geq \frac{1}{4} \quad \text{and} \quad \frac{1}{\mu_1} = 1 - \frac{1}{\mu_2} - \frac{1}{\mu_3}$$

as $\Re\left(\frac{1}{z}\right) \leq \frac{1}{2}$ for all $z \in R_D \setminus \{0, 1\}$ by Claim 29. Moreover, there are only finitely many $z \in R_D \setminus \{0\}$ such that $\Re\left(\frac{1}{z}\right) \geq \frac{1}{4}$ since these are precisely the elements of

$R_D \setminus \{0\}$ contained in the disk with center 2 and radius 2 and R_D forms a discrete subset of \mathbb{C} . Therefore, there are only finitely many unordered triples μ_1, μ_2, μ_3 of elements of $R_D \setminus \{0, 1\}$ such that $\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} = 1$. If $D = 1$, then there are exactly 23 such unordered triples (see Figure 1); if $D = 2$, there are 9; if $D = 3$, there are 27 (see Figure 2); if $D = 7$, there are 14; if $D = 11$, there are 3; if $D = 15$, there are 5. In the other cases, 2, 3 and 4 are the only elements $z \in R_D \setminus \{0, 1\}$ such that $\Re\left(\frac{1}{z}\right) \geq \frac{1}{4}$ by Claim 30, and it follows that the only triples (μ_1, μ_2, μ_3) of elements of $R_D \setminus \{0, 1\}$ such that $\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} = 1$ are $(2, 3, 6)$, $(2, 4, 4)$ and $(3, 3, 3)$ up to permutation (see Figure 3 and Figure 4). Therefore, there are only finitely many possible values for the triple $(\lambda_1, \lambda_2, \lambda_3)$, and these are $(-5, -2, -1)$, $(-3, -3, -1)$ and $(-2, -2, -2)$ up to permutation if D is different from 1, 2, 3, 7, 11 and 15. If $\lambda_1, \lambda_2, \lambda_3$ equal $-5, -2, -1$, then we have

$$M_4^f(\lambda) = \lambda^3 - 159\lambda^2 + 7419\lambda - 84221,$$

which does not split into linear factors in $R_D[\lambda]$ since it is irreducible over \mathbb{Q} of degree 3 and R_D is contained in an extension of \mathbb{Q} of degree 2. If $\lambda_1, \lambda_2, \lambda_3$ equal $-3, -3, -1$, then we have

$$M_5^f(\lambda) = (\lambda^3 + 267\lambda^2 + 20871\lambda + 414157)^2,$$

which does not split into linear factors in $R_D[\lambda]$ either since it is the square of an irreducible polynomial over \mathbb{Q} of degree 3 and R_D is contained in an extension of \mathbb{Q} of degree 2. Therefore, since the polynomials M_n^f , with $n \in \{3, 4, 5\}$, split into linear factors in $R_D[\lambda]$ by Corollary 20, we have

$$(\lambda_1, \lambda_2, \lambda_3) \in \begin{cases} \{(-4, -1 - i, -1 + i), (-2, -2, -2), \\ (-1 - i, -1 - i, 2i), (-1 + i, -1 + i, -2i)\} & \text{if } D = 1 \\ \{(-2, -2, -2), (-2, -i\sqrt{2}, i\sqrt{2})\} & \text{if } D = 2 \\ \left\{(-2, -2, -2), \left(\frac{-3-i\sqrt{7}}{2}, \frac{-3-i\sqrt{7}}{2}, \frac{-1+i\sqrt{7}}{2}\right), \right. \\ \left. \left(\frac{-3+i\sqrt{7}}{2}, \frac{-3+i\sqrt{7}}{2}, \frac{-1-i\sqrt{7}}{2}\right), \left(\frac{-1-i\sqrt{7}}{2}, \frac{-1-i\sqrt{7}}{2}, \frac{1+i\sqrt{7}}{2}\right), \right. \\ \left. \left(\frac{-1+i\sqrt{7}}{2}, \frac{-1+i\sqrt{7}}{2}, \frac{1-i\sqrt{7}}{2}\right)\right\} & \text{if } D = 7 \\ \{(-2, -2, -2)\} & \text{otherwise} \end{cases}$$

up to permutation (see Table 2, Table 3 and Table 4). If $\lambda_1, \lambda_2, \lambda_3$ equal $-2, -2, -2$, then f is conjugate to $z \mapsto \frac{1}{z^2}$; in the other cases, f is a Lattès map by Corollary 27. Thus, the lemma is proved. \square

We shall now prove the two facts below, which are used in our proof of Lemma 28.

Claim 29. Suppose that D is a positive squarefree integer. Then $\Re\left(\frac{1}{z}\right) \leq \frac{1}{2}$ for all $z \in R_D \setminus \{0, 1\}$.

Proof. Assume that $z \in R_D \setminus \{0\}$ satisfies $\Re\left(\frac{1}{z}\right) > \frac{1}{2}$, and let us prove that $z = 1$. We have

$$2\Re(z) = 2\Re\left(\frac{1}{z}\right)N(z) > N(z),$$

which yields $2\Re(z) \geq N(z) + 1$ since $2\Re(z)$ and $N(z)$ are integers, and hence

$$\text{disc}_T((T - z)(T - \bar{z})) = (2\Re(z))^2 - 4N(z) \geq (N(z) - 1)^2 \geq 0,$$

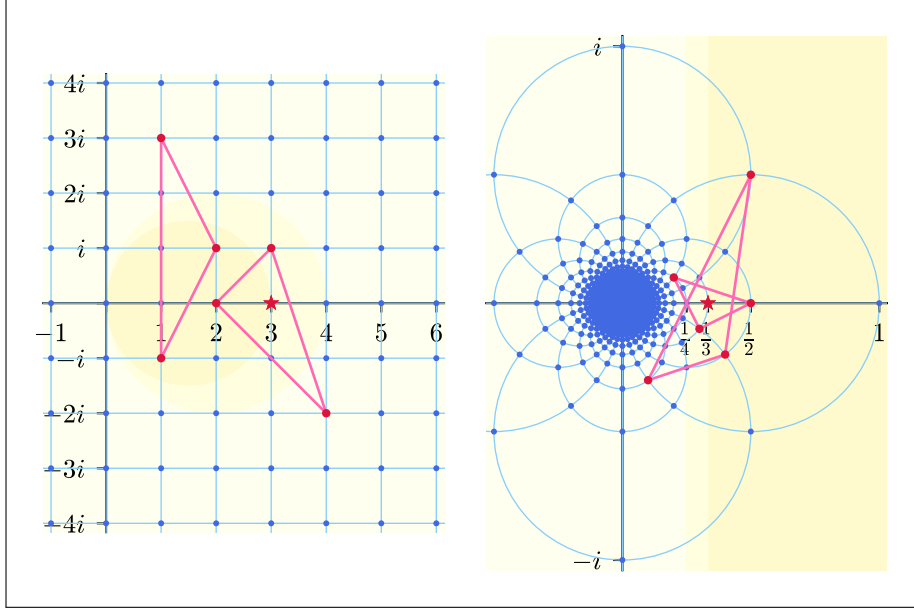


FIGURE 1. Left: The lattice R_1 and 3 of the 23 unordered triples μ_1, μ_2, μ_3 of elements of $R_1 \setminus \{0, 1\}$ such that $\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} = 1$. Right: The inversion of R_1 and of these triples. If μ_1, μ_2, μ_3 is such a triple, then, up to relabeling, we have $\Re\left(\frac{1}{\mu_3}\right) \geq \frac{1}{3}$, $\Re\left(\frac{1}{\mu_2}\right) \geq \frac{1}{4}$ and $\frac{1}{3}$ is the centroid of the triangle with vertices $\frac{1}{\mu_1}, \frac{1}{\mu_2}, \frac{1}{\mu_3}$.

TABLE 2. Decomposition of M_3^f into irreducible factors in $R_D[\lambda]$ for all the unordered triples $\lambda_1, \lambda_2, \lambda_3$ of elements of $\mathbb{C} \setminus \{0, 1\}$ – up to complex conjugation – such that M_1^f and M_2^f split into linear factors in $R_D[\lambda]$ but M_3^f does not.

D	$\lambda_1, \lambda_2, \lambda_3$	Factorization of M_3^f in $R_D[\lambda]$
1	$-3 - 2i, -2 + i, -1$	$\lambda^2 + (22 + 4i)\lambda + 121 + 40i$
1	$-1 - 4i, -1, -1 + i$	$\lambda^2 + (10 + 12i)\lambda + 5 + 48i$
1	$-1 - i, -3i, i$	$\lambda^2 + (12 + 2i)\lambda + 15 - 28i$
1	$-1, -i, 1 + 2i$	$\lambda^2 + (-2 - 4i)\lambda + 25 + 8i$
2	$-1 - 2i\sqrt{2}, -1, -1 + i\sqrt{2}$	$\lambda^2 + (10 + 4i\sqrt{2})\lambda + 33 + 16i\sqrt{2}$
3	$-3 - 2i\sqrt{3}, \frac{-3+i\sqrt{3}}{2}, -1$	$\lambda^2 + (20 + 6i\sqrt{3})\lambda + 79 + 54i\sqrt{3}$
3	$-2 - i\sqrt{3}, -2 + i\sqrt{3}, -1$	$\lambda^2 + 18\lambda + 89$
3	$-1, -i\sqrt{3}, i\sqrt{3}$	$\lambda^2 + 2\lambda + 25$
7	$\frac{-5-i\sqrt{7}}{2}, \frac{-5+i\sqrt{7}}{2}, -1$	$\lambda^2 + 22\lambda + 125$
7	$-2 - i\sqrt{7}, \frac{-3+i\sqrt{7}}{2}, -1$	$\lambda^2 + (16 + 2i\sqrt{7})\lambda + 67 + 14i\sqrt{7}$
7	$-1, \frac{-1-i\sqrt{7}}{2}, i\sqrt{7}$	$\lambda^2 + (4 - 2i\sqrt{7})\lambda + 19 - 2i\sqrt{7}$
15	$\frac{-3-i\sqrt{15}}{2}, \frac{-3+i\sqrt{15}}{2}, -1$	$\lambda^2 + 14\lambda + 61$
15	$-1, \frac{-1-i\sqrt{15}}{2}, \frac{-1+i\sqrt{15}}{2}$	$\lambda^2 + 6\lambda + 29$

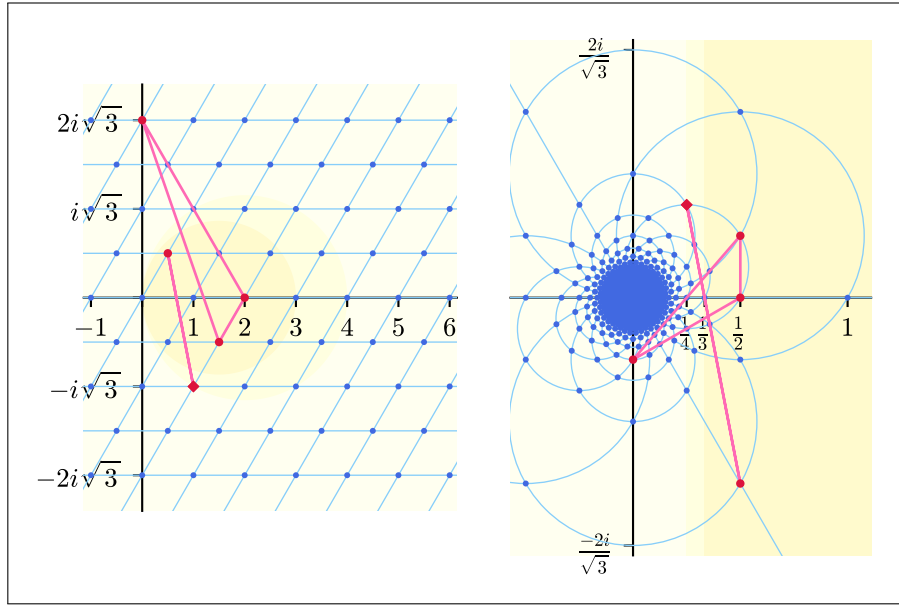


FIGURE 2. Left: The lattice R_3 and 2 of the 27 unordered triples μ_1, μ_2, μ_3 of elements of $R_3 \setminus \{0, 1\}$ such that $\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} = 1$. Right: The inversion of R_3 and of these triples.

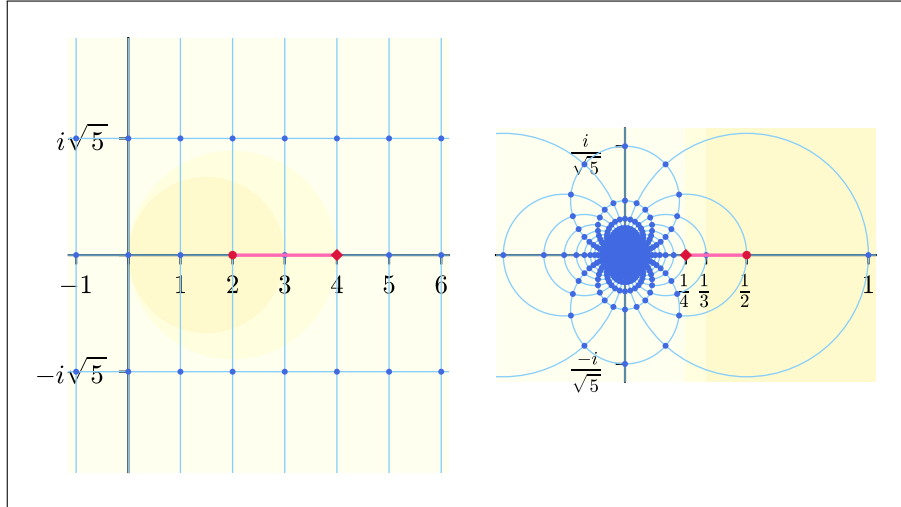


FIGURE 3. Left: The lattice R_5 and 1 of the 3 unordered triples μ_1, μ_2, μ_3 of elements of $R_5 \setminus \{0, 1\}$ such that $\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} = 1$. Right: The inversion of R_5 and of this triple. The only elements $z \in R_5 \setminus \{0, 1\}$ such that $\Re\left(\frac{1}{z}\right) \geq \frac{1}{4}$ are 2, 3 and 4.

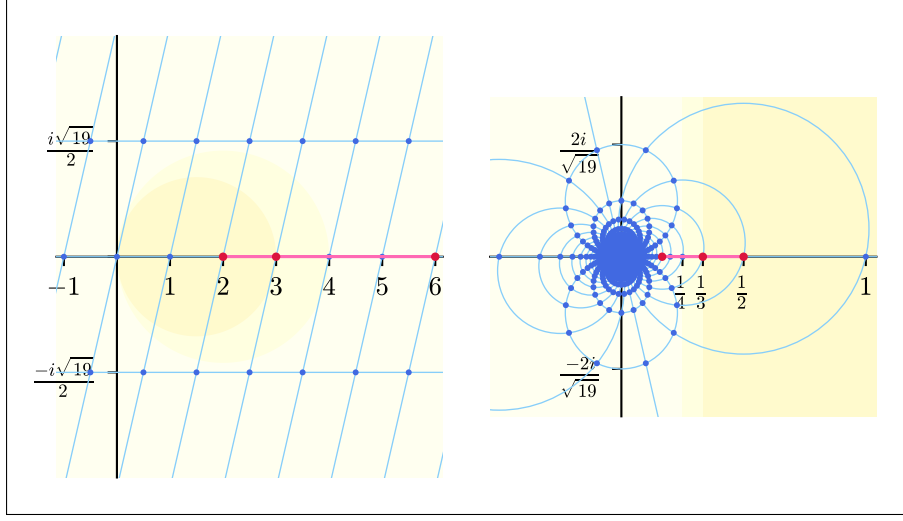


FIGURE 4. Left: The lattice R_{19} and 1 of the 3 unordered triples μ_1, μ_2, μ_3 of elements of $R_{19} \setminus \{0, 1\}$ such that $\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} = 1$. Right: The inversion of R_{19} and of this triple.

where disc_T denotes the discriminant with respect to T . Therefore, we have $z \in \mathbb{R}$, which yields $z \in \mathbb{Z} \setminus \{0\}$ since $R_D \cap \mathbb{R} = \mathbb{Z}$, and hence $z = 1$ since $\frac{1}{z} > \frac{1}{2}$. Thus, the claim is proved. \square

Claim 30. Suppose that D is a positive squarefree integer different from 1, 2, 3, 7, 11 and 15 and $z \in R_D \setminus \{0\}$ satisfies $\Re\left(\frac{1}{z}\right) \geq \frac{1}{4}$. Then $z \in \{1, 2, 3, 4\}$.

Proof. There exists $(x, y) \in \mathbb{Z}^2 \setminus \{0\}$ such that $z = x + y\alpha_D$. If $D \equiv 1, 2 \pmod{4}$ and $z \in R_D \setminus \mathbb{Z}$, then $D \geq 5$ and $|y| \geq 1$, and hence

$$\Re\left(\frac{1}{z}\right) = \frac{x}{x^2 + Dy^2} \leq \max_{t \in \mathbb{R}} \frac{t}{t^2 + 5} = \frac{1}{2\sqrt{5}} < \frac{1}{4},$$

which contradicts our hypothesis. If $D \equiv 3 \pmod{4}$ and $z \in R_D \setminus \mathbb{Z}$, then $D \geq 19$ and $|y| \geq 1$, and hence

$$\Re\left(\frac{1}{z}\right) = \frac{x + \frac{1}{2}y}{\left(x + \frac{1}{2}y\right)^2 + \frac{D}{4}y^2} \leq \max_{t \in \mathbb{R}} \frac{t}{t^2 + \frac{19}{4}} = \frac{1}{\sqrt{19}} < \frac{1}{4},$$

which also contradicts our hypothesis. Therefore, we have $z \in \mathbb{Z} \setminus \{0\}$, and hence $z \in \{1, 2, 3, 4\}$ since $\frac{1}{z} \geq \frac{1}{4}$. Thus, the claim is proved. \square

By Lemma 28, we are reduced to studying the quadratic rational maps that have a superattracting or multiple fixed point.

Lemma 31. *Assume that D is a positive squarefree integer and $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a quadratic rational map that has a superattracting fixed point and whose multiplier at each cycle with period less than or equal to 4 lies in R_D . Then f is either a power map or a Chebyshev map.*

TABLE 3. Decomposition of M_4^f into irreducible factors in $R_D[\lambda]$ for all the unordered triples $\lambda_1, \lambda_2, \lambda_3$ of elements of $\mathbb{C} \setminus \{0, 1\}$ other than $-5, -2, -1$ – up to complex conjugation – such that M_n^f splits into linear factors in $R_D[\lambda]$, with $n \in \{1, 2, 3\}$, but M_4^f does not.

D	$\lambda_1, \lambda_2, \lambda_3$	Factorization of M_4^f in $R_D[\lambda]$
1	$-2 - i, -2i, i$	$(\lambda - 1)(\lambda^2 + (6 + 12i)\lambda + 41 + 60i)$
1	$-1 - 2i, -1, -1 + 2i$	$(\lambda - 11)(\lambda^2 + 12\lambda + 211)$
2	$-2, -1 - i\sqrt{2}, -1 + i\sqrt{2}$	$(\lambda - 1)(\lambda^2 + 2\lambda + 37)$
3	$\frac{-7-i\sqrt{3}}{2}, -1 - i\sqrt{3}, \frac{-1+i\sqrt{3}}{2}$	$\lambda^3 + \frac{99-3i\sqrt{3}}{2}\lambda^2 + \frac{1449+9i\sqrt{3}}{2}\lambda + 4267 + 768i\sqrt{3}$
3	$-2 - 2i\sqrt{3}, \frac{-3-i\sqrt{3}}{2}, \frac{-1+i\sqrt{3}}{2}$	$\left(\lambda + \frac{1+7i\sqrt{3}}{2}\right) \left(\lambda^2 + (5 - i\sqrt{3})\lambda + \frac{95-17i\sqrt{3}}{2}\right)$
3	$-2 - i\sqrt{3}, \frac{-1-i\sqrt{3}}{2}, i\sqrt{3}$	$(\lambda + 8 + 5i\sqrt{3})(\lambda^2 + (-19 - i\sqrt{3})\lambda - 62 + 65i\sqrt{3})$
3	$-2, \frac{-1-3i\sqrt{3}}{2}, \frac{-1+i\sqrt{3}}{2}$	$\lambda^3 + \frac{39-7i\sqrt{3}}{2}\lambda^2 + \frac{261-19i\sqrt{3}}{2}\lambda + 449 - 302i\sqrt{3}$
3	$-1, \frac{-1-i\sqrt{3}}{2}, 1 + 2i\sqrt{3}$	$\lambda^3 + (33 - 12i\sqrt{3})\lambda^2 + (-297 - 132i\sqrt{3})\lambda + 103 + 1392i\sqrt{3}$
3	$\frac{-1-i\sqrt{3}}{2}, \frac{1+i\sqrt{3}}{2}, 1 - i\sqrt{3}$	$\lambda^3 + \frac{27+27i\sqrt{3}}{2}\lambda^2 + \frac{-423+39i\sqrt{3}}{2}\lambda + 883 + 624i\sqrt{3}$
7	$-3, \frac{-1-i\sqrt{7}}{2}, \frac{-1+i\sqrt{7}}{2}$	$\lambda^3 + 25\lambda^2 + 187\lambda + 587$
7	$-1, \frac{1-i\sqrt{7}}{2}, \frac{1+i\sqrt{7}}{2}$	$\lambda^3 + \lambda^2 - 5\lambda - 413$

Proof. There exists a parameter $c \in \mathbb{C}$ such that f is conjugate to $f_c: z \mapsto z^2 + c$. Let us prove that $c \in \{-2, 0\}$. By Corollary 20, the polynomials

$$M_1^{f_c}(\lambda) = \lambda^3 - 2\lambda^2 + 4c\lambda \quad \text{and} \quad M_3^{f_c}(\lambda) = \lambda^2 + (-8c - 16)\lambda + 64c^3 + 128c^2 + 64c + 64$$

split into linear factors in $R_D[\lambda]$, and hence $4c$ lies in R_D and the discriminants

$$\text{disc } M_1^{f_c} = -2^2(4c - 1)(4c)^2 \quad \text{and} \quad \text{disc } M_3^{f_c} = -2^2(4c + 7)(4c)^2$$

are squares in R_D . Therefore, we have $c = 0$ or there exist $a, b \in R_D$ such that

$$-(4c - 1) = a^2 \quad \text{and} \quad -(4c + 7) = b^2.$$

In the latter case, we have $(a - b)(a + b) = 8$, which yields

$$a = \frac{(a - b)^2 + 8}{2(a - b)} \in R_D \cap \left\{ \frac{e^2 + 8}{2e} : e \in R_D \text{ and } N(e) \text{ divides } 64 \right\},$$

TABLE 4. Decomposition of M_5^f into irreducible factors in $R_D[\lambda]$ for all the unordered triples $\lambda_1, \lambda_2, \lambda_3$ of elements of $\mathbb{C} \setminus \{0, 1\}$ other than $-3, -3, -1$ – up to complex conjugation – such that M_n^f splits into linear factors in $R_D[\lambda]$, with $n \in \{1, \dots, 4\}$, but M_5^f does not.

D	$\lambda_1, \lambda_2, \lambda_3$	Factorization of M_5^f in $R_D[\lambda]$
1	$-2 - i, -2 - i, -1 + i$	$(\lambda^3 + (10 + 23i)\lambda^2 + (33 + 188i)\lambda + 758 + 1703i)^2$
1	$-1 - 2i, -1 - 2i, i$	$(\lambda^3 + (-5 + 32i)\lambda^2 + (-633 - 640i)\lambda + 605 - 11584i)^2$
1	$-i, -i, 1 + i$	$(\lambda^3 + (4 + 31i)\lambda^2 + (-171 - 176i)\lambda - 700 + 1699i)^2$
2	$-1 - i\sqrt{2}, -1 - i\sqrt{2}, i\sqrt{2}$	$(\lambda^3 + (3 + 3i\sqrt{2})\lambda^2 + (-27 - 42i\sqrt{2})\lambda + 3 - 343i\sqrt{2})^2$
3	$-2 - i\sqrt{3}, -2 - i\sqrt{3}, \frac{-1+i\sqrt{3}}{2}$	$(\lambda^3 + (-12 - 21i\sqrt{3})\lambda^2 + (-573 + 36i\sqrt{3})\lambda - 8380 + 2709i\sqrt{3})^2$
3	$\frac{-3-i\sqrt{3}}{2}, \frac{-3-i\sqrt{3}}{2}, -1 + i\sqrt{3}$	$(\lambda^3 + \frac{15-5i\sqrt{3}}{2}\lambda^2 + \frac{-87-169i\sqrt{3}}{2}\lambda - 320 - 709i\sqrt{3})^2$
3	$\frac{-1-i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}, 1 + i\sqrt{3}$	$(\lambda^3 + \frac{3-3i\sqrt{3}}{2}\lambda^2 + \frac{-147+45i\sqrt{3}}{2}\lambda - 577 - 720i\sqrt{3})^2$
3	$-i\sqrt{3}, -i\sqrt{3}, \frac{1+i\sqrt{3}}{2}$	$(\lambda^3 + (42 - 29i\sqrt{3})\lambda^2 + (-1329 - 232i\sqrt{3})\lambda - 7742 + 4897i\sqrt{3})^2$

and hence

$$a \in \begin{cases} \{-3, -2, -i, i, 2, 3\} & \text{if } D = 1 \\ \{-3, 0, 3\} & \text{if } D = 2 \\ \{-3, \frac{-3-i\sqrt{3}}{2}, \frac{-3+i\sqrt{3}}{2}, \frac{3-i\sqrt{3}}{2}, \frac{3+i\sqrt{3}}{2}, 3\} & \text{if } D = 3 \\ \{-3, -1, 1, 3\} & \text{if } D = 7 \\ \{-3, 3\} & \text{otherwise} \end{cases} .$$

Therefore, in the latter case, we have

$$c = \frac{1 - a^2}{4} \in \begin{cases} \{-2, \frac{-3}{4}, \frac{1}{2}\} & \text{if } D = 1 \\ \{-2, \frac{1}{4}\} & \text{if } D = 2 \\ \{-2, \frac{-1-3i\sqrt{3}}{8}, \frac{-1+3i\sqrt{3}}{8}\} & \text{if } D = 3 \\ \{-2, 0\} & \text{if } D = 7 \\ \{-2\} & \text{otherwise} \end{cases} ,$$

and hence $c \in \{-2, 0\}$ since the polynomial $M_4^{f_c}$ splits into linear factors in $R_D[\lambda]$ by Corollary 20 (see Table 5). Thus, the lemma is proved. \square

TABLE 5. Decomposition of $M_4^{f^c}$ into irreducible factors in $R_D[\lambda]$ for the values of D and c appearing in our proof of Lemma 31.

D	c	Factorization of $M_4^{f^c}$ in $R_D[\lambda]$
1	$\frac{-3}{4}$	$\lambda^3 - 39\lambda^2 + 939\lambda - 5221$
1	$\frac{1}{2}$	$\lambda^3 - 44\lambda^2 + 784\lambda - 8896$
2	$\frac{1}{4}$	$\lambda^3 - 47\lambda^2 + 779\lambda - 4861$
3	$\frac{-1-3i\sqrt{3}}{8}$	$\lambda^3 + \frac{-109+3i\sqrt{3}}{2}\lambda^2 + \frac{1177+15i\sqrt{3}}{2}\lambda - 2983 - 1218i\sqrt{3}$
3	$\frac{-1+3i\sqrt{3}}{8}$	$\lambda^3 + \frac{-109-3i\sqrt{3}}{2}\lambda^2 + \frac{1177-15i\sqrt{3}}{2}\lambda - 2983 + 1218i\sqrt{3}$

By Lemma 28 and Lemma 31, it remains to examine the quadratic rational maps that have a multiple fixed point and whose multipliers lie in the ring of integers of a given imaginary quadratic field. We prove that there is no such map.

Lemma 32. *Assume that D is a positive squarefree integer and $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a quadratic rational map whose multiplier at each cycle with period less than or equal to 5 lies in R_D . Then the fixed points for f are all simple.*

Proof. To obtain a contradiction, suppose that f has a multiple fixed point. If f has a unique fixed point, then f is conjugate to h by Proposition 21, and hence

$$M_5^h(\lambda) = (\lambda^3 - 309\lambda^2 + 27399\lambda - 696691)^2$$

splits into linear factors in $R_D[\lambda]$ by Corollary 20, which is impossible since it is the square of an irreducible polynomial over \mathbb{Q} of degree 3 and R_D is contained in an extension of \mathbb{Q} of degree 2. Thus, f has exactly two fixed points, and it follows that f is conjugate to $g_{a,1}$ by Proposition 21, where $a \in R_D \setminus \{1\}$ is the multiplier of f at its simple fixed point. By Corollary 20, the polynomial

$$M_3^{g_{a,1}}(\lambda) = \lambda^2 + (-4a^2 - 16a - 18)\lambda + 36a^3 + 112a^2 + 124a + 89$$

splits into linear factors in $R_D[\lambda]$, and hence its discriminant

$$\text{disc } M_3^{g_{a,1}} = 2^4(a+2)(a-1)^3$$

is a square in R_D . It follows that there exists $b \in R_D$ such that $(a-1)(a+2) = b^2$, and we have

$$(2a - 2b + 1)(2a + 2b + 1) = 9.$$

Therefore, we have

$$a = \frac{(2a - 2b + 1)^2 - 2(2a - 2b + 1) + 9}{4(2a - 2b + 1)},$$

which yields

$$a \in (R_D \setminus \{1\}) \cap \left\{ \frac{c^2 - 2c + 9}{4c} : c \in R_D \text{ and } N(c) \text{ divides } 81 \right\},$$

and hence

$$a \in \begin{cases} \{-3, -2, -1, 0, 2\} & \text{if } D = 2 \\ \{-3, -2, \frac{-1-i\sqrt{3}}{2}, \frac{-1+i\sqrt{3}}{2}, 2\} & \text{if } D = 3 \\ \{-3, -2, 2\} & \text{otherwise} \end{cases}.$$

TABLE 6. Decomposition of $M_4^{g^{a,1}}$ into irreducible factors in $R_D[\lambda]$ for the values of D and a appearing in our proof of Lemma 32.

D	a	Factorization of $M_4^{g^{a,1}}$ in $R_D[\lambda]$
2	-1	$\lambda^3 - 15\lambda^2 + 255\lambda - 1457$
2	0	$\lambda^3 - 47\lambda^2 + 779\lambda - 4861$
3	$\frac{-1-i\sqrt{3}}{2}$	$\lambda^3 + (-21 + 14i\sqrt{3})\lambda^2 + (99 - 124i\sqrt{3})\lambda - 1279 + 542i\sqrt{3}$
3	$\frac{-1+i\sqrt{3}}{2}$	$\lambda^3 + (-21 - 14i\sqrt{3})\lambda^2 + (99 + 124i\sqrt{3})\lambda - 1279 - 542i\sqrt{3}$

Note that the polynomial

$$M_4^{g^{-3,1}}(\lambda) = (\lambda - 31)(\lambda^2 + 80\lambda + 1231)$$

does not split into linear factors in $R_D[\lambda]$ since it has two non-integer real roots. Moreover, the polynomials

$$M_4^{g^{-2,1}}(\lambda) = \lambda^3 + 9\lambda^2 + 123\lambda + 1307 \quad \text{and} \quad M_4^{g^{2,1}}(\lambda) = \lambda^3 - 231\lambda^2 + 17211\lambda - 407861$$

do not split into linear factors in $R_D[\lambda]$ either since they are irreducible over \mathbb{Q} of degree 3 and R_D is contained in an extension of \mathbb{Q} of degree 2. This contradicts the fact that $M_4^{g^{a,1}}$ splits into linear factors in $R_D[\lambda]$ by Corollary 20 (see Table 6). Thus, the lemma is proved. \square

Finally, we have proved Theorem 9, which follows immediately from Lemma 28, Lemma 31 and Lemma 32.

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