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SIMULTANEOUSLY PREPERIODIC INTEGERS FOR QUADRATIC POLYNOMIALS

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Abstract. In this article, we study the set of parameters $c \in \mathbb{C}$ for which two given complex numbers a and b are simultaneously preperiodic for the quadratic polynomial $f_c(z) = z^2 + c$. Combining complex-analytic and arithmetic arguments, Baker and DeMarco showed that this set of parameters is infinite if and only if $a^2 = b^2$. Recently, Buff answered a question of theirs, proving that the set of parameters $c \in \mathbb{C}$ for which both 0 and 1 are preperiodic for f_c is equal to $\{-2, -1, 0\}$. Following his approach, we complete the description of these sets when a and b are two given integers with $|a| \neq |b|$.

1. Introduction

For $c \in \mathbb{C}$, let $f_c \colon \mathbb{C} \to \mathbb{C}$ be the complex quadratic map

$$f_c \colon z \mapsto z^2 + c\,.$$

Given a point $z \in \mathbb{C}$, we study the sequence $\left(f_c^{\circ n}(z)\right)_{n \geq 0}$ of iterates of f_c at z. The set $\{f_c^{\circ n}(z) : n \geq 0\}$ is called the *forward orbit* of z under f_c.

The point z is said to be *periodic* for f_c if there exists an integer $p \geq 1$ such that $f_c^{\circ p}(z) = z$. The least such integer p is called the *period* of z. The point z is said to be *preperiodic* for f_c if its forward orbit is finite or, equivalently, if there is an integer $k \geq 0$ such that $f_c^{\circ k}(z)$ is periodic for f_c. The smallest integer k with this property is called the *preperiod* of z.

Definition 1. For $a \in \mathbb{C}$, let $\mathcal{S}_a$ be the set defined by

$$\mathcal{S}_a = \{c \in \mathbb{C} : a \text{ is preperiodic for } f_c\}\,.$$

In this paper, we wish to examine these sets of parameters.

For $n \geq 0$, let $F_n \in \mathbb{Z}[c, z]$ be the polynomial given by

$$F_n(c, z) = f_c^{\circ n}(z)\,.$$

The sequence $(F_n)_{n \geq 0}$ satisfies $F_0(c, z) = z$ and the recursion formulas

$$F_n(c, z) = F_{n-1}\left(c, z^2 + c\right) = F_{n-1}(c, z)^2 + c \quad \text{for} \quad n \geq 1\,.$$

In particular, when $n \geq 1$, the polynomial F_n is monic in c of degree 2^{n-1} and monic in z of degree 2^n.

Now, given a point $a \in \mathbb{C}$, define – for $k \geq 0$ and $p \geq 1$ – the set

$$\mathcal{S}_a^{k,p} = \{c \in \mathbb{C} : F_{k+p}(c, a) = F_k(c, a)\}\,.$$

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For all $k \geq 0$ and $p \geq 1$, the set $\mathcal{S}_a^{k,p}$ contains at most 2^{k+p-1} elements and consists of the parameters $c \in \mathbb{C}$ for which the point a is preperiodic for f_c with preperiod less than or equal to k and period dividing p.

In particular, it follows that the set

$$\mathcal{S}_a = \bigcup_{k \geq 0,\, p \geq 1} \mathcal{S}_a^{k,p}$$

is countable.

Proposition 2. *For every $a \in \mathbb{C}$, the set $\mathcal{S}_a$ is infinite.*

Proof. To obtain a contradiction, suppose that $\mathcal{S}_a$ contains finitely many elements. Then, since the sequence $\left(\mathcal{S}_a^{n,1}\right)_{n \geq 0}$ is increasing, there exists an integer $N \geq 0$ such that $\mathcal{S}_a^{n+1,1} = \mathcal{S}_a^{n,1}$ for all $n \geq N$. Now, note that, for every $n \geq 0$, we have

$$F_{n+2}(c,a) - F_{n+1}(c,a) = \left(F_{n+1}(c,a) - F_n(c,a)\right)\left(F_{n+1}(c,a) + F_n(c,a)\right) .$$

It follows that, if $n \geq N$ and γ is a root of the polynomial $F_{n+1}(c,a) + F_n(c,a)$, then

$$F_{n+1}(\gamma,a) - F_n(\gamma,a) = F_{n+1}(\gamma,a) + F_n(\gamma,a) = 0\,,$$

and hence $F_{n+1}(\gamma,a) = F_n(\gamma,a) = 0$, which yields $\gamma = 0$. Therefore, we have $F_n(0,a) = 0$ and $F_{n+1}(c,a) + F_n(c,a) = c^{2^n}$ for all $n \geq N$. In particular, we get

$$\frac{\partial\left(F_{N+2} + F_{N+1}\right)}{\partial c}(0,a) = 2\frac{\partial F_{N+1}}{\partial c}(0,a)F_{N+1}(0,a) + 2\frac{\partial F_N}{\partial c}(0,a)F_N(0,a) + 2 = 2\,,$$

which contradicts the fact that $F_{N+2}(c,a) + F_{N+1}(c,a) = c^{2^{N+1}}$. ☐

Remark 3. Note that, if $a \in \mathbb{C}$, then $f_c(a) = f_c(-a)$ for all $c \in \mathbb{C}$. Consequently, we have $\mathcal{S}_a = \mathcal{S}_{-a}$ and $\mathcal{S}_a^{k,p} = \mathcal{S}_{-a}^{k,p}$ for all $k \geq 1$ and $p \geq 1$.

Example 4. Assume that $a \in \mathbb{C}$. Then (see Figure 1) we have

$$\begin{aligned}
\mathcal{S}_a^{0,1} &= \left\{-a^2 + a\right\} ,\\
\mathcal{S}_a^{1,1} &= \left\{-a^2 - a, -a^2 + a\right\} ,\\
\mathcal{S}_a^{0,2} &= \left\{-a^2 - a - 1, -a^2 + a\right\} ,\\
\mathcal{S}_a^{1,2} &= \left\{-a^2 - a - 1, -a^2 - a, -a^2 + a - 1, -a^2 + a\right\} .
\end{aligned}$$

Here, the problem we are interested in is the description of the sets $\mathcal{S}_a \cap \mathcal{S}_b$ when a and b are two given complex numbers.

Example 5. Suppose that $a \in \mathbb{C}$. Then (see Figure 2) we have

$$-a^2 - a - 1 = -(a+1)^2 + (a+1) - 1 \in \mathcal{S}_a^{0,2} \cap \mathcal{S}_{a+1}^{1,2}$$

and

$$-a^2 - a = -(a+1)^2 + (a+1) \in \mathcal{S}_a^{1,1} \cap \mathcal{S}_{a+1}^{0,1}\,.$$

Example 6. We have $-2 \in \mathcal{S}_0^{2,1} \cap \mathcal{S}_1^{1,1}$, $-1 \in \mathcal{S}_0^{0,2} \cap \mathcal{S}_1^{1,2}$ and $0 \in \mathcal{S}_0^{0,1} \cap \mathcal{S}_1^{0,1}$ (see Figure 3).

$$\begin{tikzcd} a \arrow[loop right, "f_c"] \end{tikzcd} \qquad \text{for } c = -a^2 + a,$$

$$\begin{tikzcd} a \arrow[r, bend left, "f_c"] & -a \arrow[loop right, "f_c"] \end{tikzcd} \qquad \text{for } c = -a^2 - a,$$

$$\begin{tikzcd} a \arrow[r, bend left, "f_c"] & -a-1 \arrow[l, bend left, "f_c"] \end{tikzcd} \qquad \text{for } c = -a^2 - a - 1,$$

$$\begin{tikzcd} a \arrow[r, bend left, "f_c"] & a-1 \arrow[r, bend left, "f_c"] & -a \arrow[l, bend left, "f_c"] \end{tikzcd} \qquad \text{for } c = -a^2 + a - 1.$$

FIGURE 1. Some parameters $c \in \mathbb{C}$ for which a given complex number a is preperiodic for f_c.

$$\begin{tikzcd} a+1 \arrow[r, bend left, "f_c"] & a \arrow[r, bend left, "f_c"] & -a-1 \arrow[l, bend left, "f_c"] \end{tikzcd} \qquad \text{for } c = -a^2 - a - 1,$$

$$\begin{tikzcd} a \arrow[r, bend left, "f_c"] & -a \arrow[loop right, "f_c"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} a+1 \arrow[loop right, "f_c"] \end{tikzcd} \qquad \text{for } c = -a^2 - a.$$

FIGURE 2. Two parameters $c \in \mathbb{C}$ for which a and $a+1$ are simultaneously preperiodic for f_c when a is a given complex number.

$$\begin{tikzcd} 0 \arrow[r, bend left, "f_{-2}"] & -2 \arrow[r, bend left, "f_{-2}"] & 2 \arrow[loop right, "f_{-2}"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 1 \arrow[r, bend left, "f_{-2}"] & -1 \arrow[loop right, "f_{-2}"] \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[r, bend left, "f_{-1}"] & 0 \arrow[r, bend left, "f_{-1}"] & -1 \arrow[l, bend left, "f_{-1}"] \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[loop right, "f_0"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 1 \arrow[loop right, "f_0"] \end{tikzcd}$$

FIGURE 3. Three parameters $c \in \mathbb{C}$ for which both 0 and 1 are preperiodic for f_c.

Since the sets $\mathcal{S}_a$ are countably infinite (see Proposition 2), we may wonder whether the sets $\mathcal{S}_a \cap \mathcal{S}_b$ are infinite. This question was answered by Baker and DeMarco in [BD11]. Using potential theory and an equidistribution result for points of small height with respect to an adelic height function, they proved that the set $\mathcal{S}_a \cap \mathcal{S}_b$ is infinite if and only if $a^2 = b^2$.

As they pointed out, their proof is not effective and does not provide any estimate on the cardinality of these sets when they are finite. In their article, Baker and DeMarco conjectured that -2, -1 and 0 were the only parameters $c \in \mathbb{C}$ for which 0 and 1 are simultaneously preperiodic for f_c (see Example 6). Using localization properties of the set of parameters $c \in \mathbb{C}$ for which both 0 and 1 have bounded forward orbit under f_c and the fact that 0 is the only parameter $c \in \mathbb{C}$ that is contained in the main cardioid of the Mandelbrot set and for which 0 is preperiodic for f_c, Buff gave an elementary proof of their conjecture in [Buf18].

Following his approach, we complete the description of the sets $\mathcal{S}_a \cap \mathcal{S}_b$ when a and b are two given integers with $|a| \neq |b|$. More precisely, we prove the following theorem, which asserts that Example 5 and Example 6 present all the parameters $c \in \mathbb{C}$ for which two given distinct and non-opposite integers are simultaneously preperiodic for the polynomial f_c:

Theorem 7. *Assume that a and b are two integers with $|b| > |a|$. Then*

- *either $a = 0$, $|b| = 1$ and $\mathcal{S}_a \cap \mathcal{S}_b = \{-2, -1, 0\}$,*
- *or $a = 0$, $|b| = 2$ and $\mathcal{S}_a \cap \mathcal{S}_b = \{-2\}$,*
- *or $|a| \geq 1$, $|b| = |a| + 1$ and $\mathcal{S}_a \cap \mathcal{S}_b = \left\{-a^2 - |a| - 1, -a^2 - |a|\right\}$,*
- *or $|b| > \max\{2, |a| + 1\}$ and $\mathcal{S}_a \cap \mathcal{S}_b = \varnothing$.*

Our proof is elementary and uses only basic analytic and arithmetic arguments. In particular, the reader does not need to be familiar with complex dynamics.

In Section 2, we reprove some well-known results on the dynamics of the polynomials f_c. In Section 3, we go back to the study of the parameter space and give a proof of Theorem 7.

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2. The dynamics of the quadratic polynomials

We shall investigate here the dynamics of the quadratic maps $f_c \colon \mathbb{C} \to \mathbb{C}$.

Given a parameter $c \in \mathbb{C}$, let $\mathcal{X}_c$ be the set

$$\mathcal{X}_c = \{z \in \mathbb{C} : z \text{ is preperiodic for } f_c\} \ ,$$

and, for $k \geq 0$ and $p \geq 1$, let $\mathcal{X}_c^{k,p}$ be the set

$$\mathcal{X}_c^{k,p} = \{z \in \mathbb{C} : F_{k+p}(c, z) = F_k(c, z)\} \ .$$

For all $k \geq 0$ and $p \geq 1$, the set $\mathcal{X}_c^{k,p}$ contains at most 2^{k+p} elements, is invariant under f_c and consists of the preperiodic points for f_c with preperiod less than or equal to k and period dividing p. In particular, we have

$$\mathcal{X}_c = \bigcup_{k \geq 0,\, p \geq 1} \mathcal{X}_c^{k,p} \ .$$

Moreover, the set $\mathcal{X}_c$ is completely invariant under f_c – that is, for every $z \in \mathbb{C}$, $f_c(z) \in \mathcal{X}_c$ if and only if $z \in \mathcal{X}_c$.

Remark 8. Note that, if $c \in \mathbb{C}$, then $f_c(z) = f_c(-z)$ for all $z \in \mathbb{C}$. Therefore, the sets $\mathcal{X}_c$ and $\mathcal{X}_c^{k,p}$, with $k \geq 1$ and $p \geq 1$, are symmetric with respect to the origin.

Proposition 9. *For every $c \in \mathbb{C}$, we have*

$$\mathcal{X}_c \subset \bigcap_{n \geq 0} \left\{ z \in \mathbb{C} : \left| f_c^{\circ n}(z) \right| \leq \rho_c \right\} ,$$

where $\rho_c = \frac{1+\sqrt{1+4|c|}}{2}$.

Proof. For every $z \in \mathbb{C}$, we have $|f_c(z)| \geq |z|^2 - |c|$, and $|z|^2 - |c| > |z|$ if and only if $|z| > \rho_c$. It follows by induction that, if $z \in \mathbb{C}$ satisfies $|z| > \rho_c$, then $\left| f_c^{\circ(k+p)}(z) \right| > \left| f_c^{\circ k}(z) \right|$ for all $k \geq 0$ and $p \geq 1$, and hence z is not preperiodic for f_c. As the set $\mathcal{X}_c$ is invariant under f_c, this completes the proof of the proposition. ☐

Now, let us study the dynamics of the polynomial f_c when c is a real parameter. Suppose that $c \in \left(-\infty, \frac{1}{4}\right]$. Then the map $f_c \colon \mathbb{R} \to \mathbb{R}$ is even and strictly increasing on $\mathbb{R}_{\geq 0}$, has two fixed points $\alpha_c \leq \beta_c$ – with equality if and only if $c = \frac{1}{4}$ – given by

$$\alpha_c = \frac{1 - \sqrt{1 - 4c}}{2} \quad \text{and} \quad \beta_c = \frac{1 + \sqrt{1 - 4c}}{2}$$

and satisfies $f_c(z) > z$ for all $z \in (\beta_c, +\infty)$. In particular, we have

$$f_c\left([-\beta_c, \beta_c]\right) = [c, \beta_c]$$

and the sequence $\left(f_c^{\circ n}(z)\right)_{n \geq 0}$ diverges to $+\infty$ for all $z \in (-\infty, -\beta_c) \cup (\beta_c, +\infty)$.

It follows that, if $c \in \left[-2, \frac{1}{4}\right]$, then

$$f_c\left([-\beta_c, \beta_c]\right) \subset [-\beta_c, \beta_c] ,$$

and hence, for every $z \in \mathbb{R}$, the point z has bounded forward orbit under f_c if and only if $z \in [-\beta_c, \beta_c]$.

Remark 10. Note that, for every $c \in \mathbb{C}$, we have $\rho_c = \beta_{-|c|}$.

Let us examine more thoroughly the dynamics of the map f_c when $c \in (-\infty, -2]$. It is related to the dynamics of the shift map in the space of sign sequences.

Let $\sigma \colon \{-1, 1\}^{\mathbb{Z}_{\geq 0}} \to \{-1, 1\}^{\mathbb{Z}_{\geq 0}}$ denote the *shift map*, which sends any sequence $\boldsymbol{\varepsilon} = (\epsilon_n)_{n \geq 0}$ of ± 1 to the sequence $(\epsilon_{n+1})_{n \geq 0}$.

A sign sequence $\boldsymbol{\varepsilon}$ is said to be *periodic* with *period* $p \geq 1$ if $\sigma^{\circ p}(\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon}$ and p is the least such integer. The sequence $\boldsymbol{\varepsilon}$ is said to be *preperiodic* with *preperiod* $k \geq 0$ if the sequence $\sigma^{\circ k}(\boldsymbol{\varepsilon})$ is periodic and k is minimal with this property.

For $k \geq 0$ and $p \geq 1$, define

$$\boldsymbol{\Sigma}^{k,p} = \left\{ \boldsymbol{\varepsilon} \in \{-1, 1\}^{\mathbb{Z}_{\geq 0}} : \sigma^{\circ(k+p)}(\boldsymbol{\varepsilon}) = \sigma^{\circ k}(\boldsymbol{\varepsilon}) \right\}$$

to be the set of all preperiodic sign sequences with preperiod less than or equal to k and period dividing p, and define

$$\boldsymbol{\Sigma} = \bigcup_{k \geq 0,\, p \geq 1} \boldsymbol{\Sigma}^{k,p}$$

to be the collection of all preperiodic sign sequences. For all $k \geq 0$ and $p \geq 1$, the set $\boldsymbol{\Sigma}^{k,p}$ contains exactly 2^{k+p} elements – each of them being completely determined by the choice of its first $k + p$ terms – and is invariant under the shift map. Moreover, the set $\mathbf{\Sigma}$ is completely invariant under the shift map – that is, any sign sequence $\boldsymbol{\varepsilon}$ is preperiodic if and only if the sequence $\sigma(\boldsymbol{\varepsilon})$ is preperiodic.

Theorem 11. *For every $c \in (-\infty, -2]$, there exists a unique map*

$$\psi_c \colon \mathbf{\Sigma} \to \mathbb{R}$$

that makes the diagram below commute and satisfies $\epsilon_0 \psi_c(\boldsymbol{\varepsilon}) \geq 0$ for all $\boldsymbol{\varepsilon} \in \mathbf{\Sigma}$.

$$\begin{tikzcd} \mathbf{\Sigma} \arrow[r, "\sigma"] \arrow[d, "\psi_c"'] & \mathbf{\Sigma} \arrow[d, "\psi_c"] \\ \mathbb{R} \arrow[r, "f_c"'] & \mathbb{R} \end{tikzcd}$$

Furthermore, for every $\boldsymbol{\varepsilon} \in \mathbf{\Sigma}$, we have

$$\epsilon_0 \psi_c(\boldsymbol{\varepsilon}) \in \left[\sqrt{-\beta_c - c}, \beta_c\right] ,$$

for all $c \in (-\infty, -2]$, and the map $\zeta_{\boldsymbol{\varepsilon}} \colon (-\infty, -2] \to \mathbb{R}$ defined by

$$\zeta_{\boldsymbol{\varepsilon}}(c) = \psi_c(\boldsymbol{\varepsilon})$$

is continuous.

Before proving Theorem 11, observe that $c \leq -\beta_c$ for all $c \in (-\infty, -2]$, with equality if and only if $c = -2$. Consequently, for $c \in (-\infty, -2]$ and $\epsilon = \pm 1$, the partial inverse $g_c^\epsilon \colon [c, +\infty) \to \mathbb{R}$ of f_c given by

$$g_c^\epsilon(z) = \epsilon \sqrt{z - c}$$

is well defined on $[-\beta_c, \beta_c]$, and we have

$$g_c^\epsilon\left([-\beta_c, \beta_c]\right) = \left[\epsilon \sqrt{-\beta_c - c}, \epsilon \beta_c\right] \subset [-\beta_c, \beta_c] \ .$$

Lemma 12. *For all $c \in (-\infty, -2]$ and all $\boldsymbol{\varepsilon} = (\epsilon_0, \ldots, \epsilon_{p-1}) \in \{-1, 1\}^p$, with $p \geq 1$, the map $g_c^{\boldsymbol{\varepsilon}} \colon [-\beta_c, \beta_c] \to [-\beta_c, \beta_c]$ defined by*

$$g_c^{\boldsymbol{\varepsilon}}(z) = g_c^{\epsilon_0} \circ \cdots \circ g_c^{\epsilon_{p-1}}(z)$$

has a unique fixed point $\mathfrak{z}_{\boldsymbol{\varepsilon}}(c)$.

Moreover, for every finite sequence $\boldsymbol{\varepsilon}$ of ± 1, the map $c \mapsto \mathfrak{z}_{\boldsymbol{\varepsilon}}(c)$ is continuous.

Claim 13. If $c \in (-\infty, -2]$, $\boldsymbol{\varepsilon} \in \{-1, 1\}^p$, with $p \geq 1$, and $\mathfrak{z}$ is a fixed point of $g_c^{\boldsymbol{\varepsilon}}$, then $\mathfrak{z} \in \mathcal{X}_c^{0,p}$ and $\epsilon_j f_c^{\circ j}(\mathfrak{z}) > 0$ for all $j \in \{0, \ldots, p-1\}$.

Proof of Claim 13. We have $f_c^{\circ p}(\mathfrak{z}) = \mathfrak{z}$ and the set $\mathcal{X}_c^{0,p}$ is invariant under f_c. Therefore, for all $j \in \{0, \ldots, p-1\}$, we have

$$f_c^{\circ j}(\mathfrak{z}) = g_c^{\epsilon_j} \circ \cdots \circ g_c^{\epsilon_{p-1}}(\mathfrak{z}) \in g_c^{\epsilon_j}\left([-\beta_c, \beta_c]\right) \cap \mathcal{X}_c^{0,p} ,$$

which yields

$$\epsilon_j f_c^{\circ j}(\mathfrak{z}) \in \left(\sqrt{-\beta_c - c}, \beta_c\right] \subset \mathbb{R}_{>0}$$

since $\epsilon_j \sqrt{-\beta_c - c}$ is preperiodic for f_c with preperiod 2 and period 1. ☐

Proof of Lemma 12. Fix $c \in (-\infty, -2]$ and $p \geq 1$. For every $\boldsymbol{\varepsilon} \in \{-1, 1\}^p$, the map $g_c^{\boldsymbol{\varepsilon}}$ has a fixed point $\mathfrak{z}_{\boldsymbol{\varepsilon}}(c)$ by the intermediate value theorem. Now, note that $\mathfrak{z}_{\boldsymbol{\varepsilon}}(c)$ is not a fixed point of $g_c^{\boldsymbol{\varepsilon}'}$ whenever $\boldsymbol{\varepsilon} \neq \boldsymbol{\varepsilon}' \in \{-1, 1\}^p$ by Claim 13. Therefore, the points $\mathfrak{z}_{\boldsymbol{\varepsilon}}(c)$, with $\boldsymbol{\varepsilon} \in \{-1, 1\}^p$, are pairwise distinct, and, since $\mathcal{X}_c^{0,p}$ contains at most 2^p elements, it follows that

$$\mathcal{X}_c^{0,p} = \{\mathfrak{z}_{\boldsymbol{\varepsilon}}(c) : \boldsymbol{\varepsilon} \in \{-1, 1\}^p\} \, .$$

Thus, for every $\boldsymbol{\varepsilon} \in \{-1, 1\}^p$, $\mathfrak{z}_{\boldsymbol{\varepsilon}}(c)$ is the unique fixed point of the map $g_c^{\boldsymbol{\varepsilon}}$.

Now, fix $p \geq 1$, $\boldsymbol{\varepsilon} = (\epsilon_0, \ldots, \epsilon_{p-1}) \in \{-1, 1\}^p$ and $c \in (-\infty, -2]$. It remains to verify that the map $c' \mapsto \mathfrak{z}_{\boldsymbol{\varepsilon}}(c')$ is continuous at c. For each $c' \in (-\infty, -2]$, choose $\boldsymbol{\varepsilon}_{c'} \in \{-1, 1\}^p$ such that $\left|\mathfrak{z}_{\boldsymbol{\varepsilon}}(c) - \mathfrak{z}_{\boldsymbol{\varepsilon}_{c'}}(c')\right|$ is minimal. Then we have

$$\left|\mathfrak{z}_{\boldsymbol{\varepsilon}}(c) - \mathfrak{z}_{\boldsymbol{\varepsilon}_{c'}}(c')\right| \leq \left(\prod_{\boldsymbol{\varepsilon}' \in \{-1,1\}^p} \left|\mathfrak{z}_{\boldsymbol{\varepsilon}}(c) - \mathfrak{z}_{\boldsymbol{\varepsilon}'}(c')\right|\right)^{\frac{1}{2^p}} = \left|F_p\left(c', \mathfrak{z}_{\boldsymbol{\varepsilon}}(c)\right) - \mathfrak{z}_{\boldsymbol{\varepsilon}}(c)\right|^{\frac{1}{2^p}}$$

for all $c' \in (-\infty, -2]$, and so $\mathfrak{z}_{\boldsymbol{\varepsilon}_{c'}}(c')$ tends to $\mathfrak{z}_{\boldsymbol{\varepsilon}}(c)$ as c' approaches c. By Claim 13, it follows that, whenever c' is close enough to c, we have $\epsilon_j f_{c'}^{\circ j}\left(\mathfrak{z}_{\boldsymbol{\varepsilon}_{c'}}(c')\right) > 0$ for all $j \in \{0, \ldots, p-1\}$, which yields $\boldsymbol{\varepsilon}_{c'} = \boldsymbol{\varepsilon}$. Thus, the limit of $\mathfrak{z}_{\boldsymbol{\varepsilon}}(c')$ as c' approaches c is $\mathfrak{z}_{\boldsymbol{\varepsilon}}(c)$, and the lemma is proved. □

We may now deduce Theorem 11 from Lemma 12.

Proof of Theorem 11. Fix $c \in (-\infty, -2]$. Assume that $\psi_c \colon \boldsymbol{\Sigma} \to \mathbb{R}$ is a map that satisfies $f_c \circ \psi_c = \psi_c \circ \sigma$ and $\epsilon_0 \psi_c(\boldsymbol{\varepsilon}) \geq 0$ for all $\boldsymbol{\varepsilon} \in \boldsymbol{\Sigma}$. Then, for all $\boldsymbol{\varepsilon} \in \boldsymbol{\Sigma}$ and all $n \geq 0$, we have

$$\psi_c(\boldsymbol{\varepsilon}) = g_c^{\epsilon_0} \circ \cdots \circ g_c^{\epsilon_n}\left(\psi_c\left(\sigma^{\circ(n+1)}(\boldsymbol{\varepsilon})\right)\right) \, .$$

It follows that, if $\boldsymbol{\varepsilon}$ is a periodic sign sequence with period $p \geq 1$, then $\psi_c(\boldsymbol{\varepsilon})$ is a fixed point of the map $g_c^{\boldsymbol{\varepsilon_p}}$, where $\boldsymbol{\varepsilon_p} = (\epsilon_0, \ldots, \epsilon_{p-1}) \in \{-1, 1\}^p$, and hence $\psi_c(\boldsymbol{\varepsilon}) = \mathfrak{z}_{\boldsymbol{\varepsilon_p}}(c)$. Therefore, for every $\boldsymbol{\varepsilon} \in \boldsymbol{\Sigma}$ with preperiod $k \geq 0$ and period $p \geq 1$, we have $\psi_c(\boldsymbol{\varepsilon}) = g_c^{\boldsymbol{\varepsilon_{pp}}}\left(\mathfrak{z}_{\boldsymbol{\varepsilon_p}}(c)\right)$, where $\boldsymbol{\varepsilon_{pp}} = (\epsilon_0, \ldots, \epsilon_{k-1}) \in \{-1, 1\}^k$ and $\boldsymbol{\varepsilon_p} = (\epsilon_k, \ldots, \epsilon_{k+p-1}) \in \{-1, 1\}^p$, adopting the convention that $g_c^{\varnothing}$ denotes the identity map of $[-\beta_c, \beta_c]$. In particular, there is at most one map $\psi_c \colon \boldsymbol{\Sigma} \to \mathbb{R}$ that satisfies the conditions above.

For $\boldsymbol{\varepsilon} = (\epsilon_n)_{n \geq 0}$ a preperiodic sign sequence with preperiod $k \geq 0$ and period $p \geq 1$, define $\boldsymbol{\varepsilon_{pp}} = (\epsilon_0, \ldots, \epsilon_{k-1}) \in \{-1, 1\}^k$, $\boldsymbol{\varepsilon_p} = (\epsilon_k, \ldots, \epsilon_{k+p-1}) \in \{-1, 1\}^p$ and $\psi_c(\boldsymbol{\varepsilon}) = g_c^{\boldsymbol{\varepsilon_{pp}}}\left(\mathfrak{z}_{\boldsymbol{\varepsilon_p}}(c)\right)$. If $\boldsymbol{\varepsilon}$ is a periodic sign sequence with period $p \geq 1$, then $f_c \circ \psi_c(\boldsymbol{\varepsilon})$ is a fixed point of the map $g_c^{\sigma(\boldsymbol{\varepsilon})_{\boldsymbol{p}}}$ since $\sigma(\boldsymbol{\varepsilon})_{\boldsymbol{p}} = (\epsilon_1, \ldots, \epsilon_{p-1}, \epsilon_0)$, and hence $f_c \circ \psi_c(\boldsymbol{\varepsilon}) = \psi_c \circ \sigma(\boldsymbol{\varepsilon})$. Similarly, if $\boldsymbol{\varepsilon} \in \boldsymbol{\Sigma}$ has preperiod $k \geq 1$ and period $p \geq 1$, then $f_c \circ \psi_c(\boldsymbol{\varepsilon}) = \psi_c \circ \sigma(\boldsymbol{\varepsilon})$ since $\sigma(\boldsymbol{\varepsilon})_{\boldsymbol{pp}} = (\epsilon_1, \ldots, \epsilon_{k-1})$ and $\sigma(\boldsymbol{\varepsilon})_{\boldsymbol{p}} = \boldsymbol{\varepsilon_p}$. Moreover, for all $\boldsymbol{\varepsilon} \in \boldsymbol{\Sigma}$, we have $\psi_c(\boldsymbol{\varepsilon}) \in g_c^{\epsilon_0}\left([-\beta_c, \beta_c]\right)$, which yields

$$\epsilon_0 \psi_c(\boldsymbol{\varepsilon}) \in \left[\sqrt{-\beta_c - c}, \beta_c\right] \subset \mathbb{R}_{\geq 0} \, .$$

Thus, the map $\psi_c \colon \boldsymbol{\Sigma} \to \mathbb{R}$ so defined has the required properties.

Furthermore, for every $\boldsymbol{\varepsilon} \in \boldsymbol{\Sigma}$, the map $\zeta_{\boldsymbol{\varepsilon}} \colon c \mapsto \psi_c(\boldsymbol{\varepsilon})$ is clearly continuous. □

Remark 14. Observe that, if $c \in (-\infty, -2]$ and $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}' \in \boldsymbol{\Sigma}$ satisfy $\epsilon_0 = -\epsilon_0'$ and $\sigma(\boldsymbol{\varepsilon}) = \sigma(\boldsymbol{\varepsilon}')$, then $\psi_c(\boldsymbol{\varepsilon}) = -\psi_c(\boldsymbol{\varepsilon}')$.

Note that the proof of Theorem 11 provides explicit formulas for the maps $\zeta_{\boldsymbol{\varepsilon}}$ with $\boldsymbol{\varepsilon} \in \mathbf{\Sigma}^{k,1}$ and $k \geq 0$.

Example 15. Suppose that $\epsilon = \pm 1$. Then

- for $\boldsymbol{\varepsilon} \in \mathbf{\Sigma}^{1,1}$ given by $\epsilon_0 = \epsilon$ and $\epsilon_1 = -1$, we have
$$\zeta_{\boldsymbol{\varepsilon}} \colon c \mapsto \psi_c(\boldsymbol{\varepsilon}) = -\epsilon\alpha_c\,;$$
- for $\boldsymbol{\varepsilon} \in \mathbf{\Sigma}^{1,1}$ given by $\epsilon_0 = \epsilon$ and $\epsilon_1 = 1$, we have
$$\zeta_{\boldsymbol{\varepsilon}} \colon c \mapsto \psi_c(\boldsymbol{\varepsilon}) = \epsilon\beta_c\,;$$
- for $\boldsymbol{\varepsilon} \in \mathbf{\Sigma}^{2,1}$ given by $\epsilon_0 = \epsilon$, $\epsilon_1 = 1$ and $\epsilon_2 = -1$, we have
$$\zeta_{\boldsymbol{\varepsilon}} \colon c \mapsto \psi_c(\boldsymbol{\varepsilon}) = \epsilon\sqrt{-\alpha_c - c}\,;$$
- for $\boldsymbol{\varepsilon} \in \mathbf{\Sigma}^{2,1}$ given by $\epsilon_0 = \epsilon$, $\epsilon_1 = -1$ and $\epsilon_2 = 1$, we have
$$\zeta_{\boldsymbol{\varepsilon}} \colon c \mapsto \psi_c(\boldsymbol{\varepsilon}) = \epsilon\sqrt{-\beta_c - c}\,.$$

Proposition 16. *Assume that $c \in (-\infty, -2]$. Then we have*
$$\mathcal{X}_c^{k,p} = \psi_c\left(\mathbf{\Sigma}^{k,p}\right) \subset [-\beta_c, \beta_c]$$
for all $k \geq 0$ and $p \geq 1$ (see Figure 4).

Furthermore, if $c \in (-\infty, -2)$, then the map $\psi_c \colon \mathbf{\Sigma} \to \mathbb{R}$ is injective.

Proof. For all $n \geq 0$, we have $f_c^{\circ n} \circ \psi_c = \psi_c \circ \sigma^{\circ n}$. Consequently, $\psi_c\left(\mathbf{\Sigma}^{k,p}\right) \subset \mathcal{X}_c^{k,p}$ for all $k \geq 0$ and $p \geq 1$.

Now, suppose that $c \in (-\infty, -2)$. Then, for all $\boldsymbol{\varepsilon} \in \mathbf{\Sigma}$ and all $n \geq 0$, we have
$$\epsilon_n f_c^{\circ n}\left(\psi_c(\boldsymbol{\varepsilon})\right) \in \left[\sqrt{-\beta_c - c}, \beta_c\right] \subset \mathbb{R}_{>0}\,.$$
Therefore, the map ψ_c is injective, and, since $\mathcal{X}_c^{k,p}$ contains at most 2^{k+p} elements, it follows that $\psi_c\left(\mathbf{\Sigma}^{k,p}\right) = \mathcal{X}_c^{k,p}$, for all $k \geq 0$ and $p \geq 1$.

It remains to prove that $\mathcal{X}_{-2}^{k,p} \subset \psi_{-2}\left(\mathbf{\Sigma}^{k,p}\right)$ for all $k \geq 0$ and $p \geq 1$. Fix $k \geq 0$ and $p \geq 1$, and suppose that $z \in \mathcal{X}_{-2}^{k,p}$. Then, for all $c \in (-\infty, -2)$, we have
$$\min_{\boldsymbol{\varepsilon} \in \mathbf{\Sigma}^{k,p}} \left|z - \psi_c(\boldsymbol{\varepsilon})\right| \leq \left(\prod_{\boldsymbol{\varepsilon} \in \mathbf{\Sigma}^{k,p}} \left|z - \psi_c(\boldsymbol{\varepsilon})\right|\right)^{\frac{1}{2^{k+p}}} = \left|F_{k+p}(c, z) - F_k(c, z)\right|^{\frac{1}{2^{k+p}}}\,.$$
As the maps $\zeta_{\boldsymbol{\varepsilon}}$, with $\boldsymbol{\varepsilon} \in \mathbf{\Sigma}^{k,p}$, are continuous at -2, it follows that $z \in \psi_{-2}\left(\mathbf{\Sigma}^{k,p}\right)$. Thus, the proposition is proved. □

Remark 17. Applying Montel's theorem, it follows from Proposition 16 that, for every $c \in (-\infty, -2]$, the filled-in Julia set of f_c – that is, the set of points $z \in \mathbb{C}$ that have bounded forward orbit under f_c – is also contained in $[-\beta_c, \beta_c]$.

Note that the map ψ_{-2} is not injective. More precisely, we have the following:

Proposition 18. *For all $\boldsymbol{\varepsilon} \neq \boldsymbol{\varepsilon}' \in \mathbf{\Sigma}$, $\psi_{-2}(\boldsymbol{\varepsilon}) = \psi_{-2}(\boldsymbol{\varepsilon}')$ if and only if there exists an integer $k \geq 2$ such that $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}' \in \mathbf{\Sigma}^{k,1}$, $\epsilon_j = \epsilon'_j$ for all $j \in \{0, \ldots, k-3\}$, $\epsilon_{k-2} = -\epsilon'_{k-2}$, $\epsilon_{k-1} = \epsilon'_{k-1} = -1$ and $\epsilon_k = \epsilon'_k = 1$.*

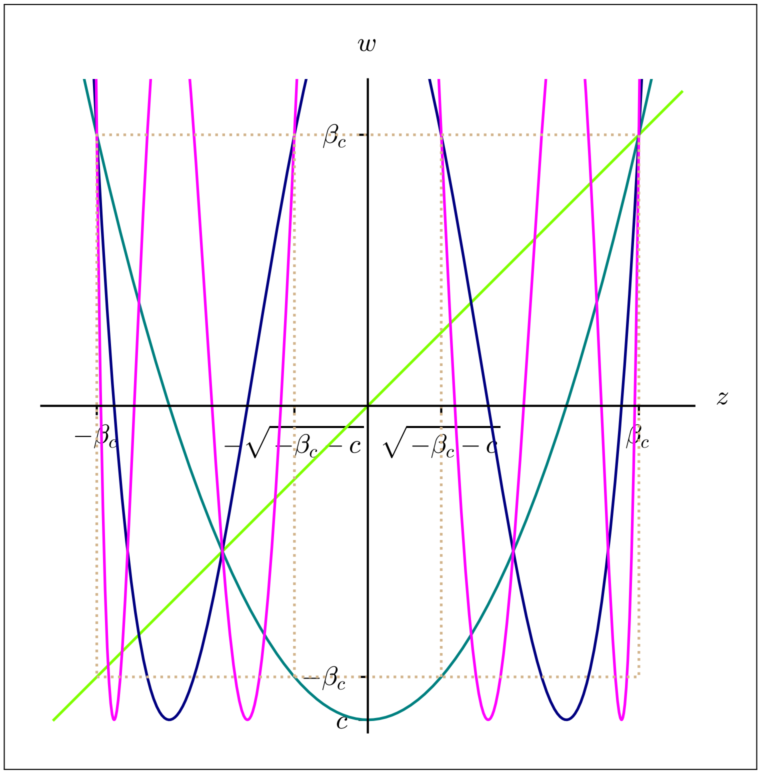

FIGURE 4. Graphs of the maps $z \mapsto F_n(c, z)$, with $n \in \{0, \ldots, 3\}$, when $c \in (-\infty, -2]$.

Proof. Suppose that $\boldsymbol{\varepsilon} \neq \boldsymbol{\varepsilon}' \in \boldsymbol{\Sigma}$ satisfy $\psi_{-2}(\boldsymbol{\varepsilon}) = \psi_{-2}(\boldsymbol{\varepsilon}')$. Then, for all $n \geq 0$, we have

$$\epsilon_n f_{-2}^{\circ n}(\psi_{-2}(\boldsymbol{\varepsilon})) \geq 0 \quad \text{and} \quad \epsilon_n' f_{-2}^{\circ n}(\psi_{-2}(\boldsymbol{\varepsilon})) \geq 0\,.$$

Since $\boldsymbol{\varepsilon} \neq \boldsymbol{\varepsilon}'$, it follows that there is an integer $k \geq 0$, which we may assume minimal, such that $f_{-2}^{\circ k}(\psi_{-2}(\boldsymbol{\varepsilon})) = 0$. For all $j \in \{0, \ldots, k-1\}$, the inequalities above are strict, and hence $\epsilon_j = \epsilon_j'$. Moreover, we have $f_{-2}^{\circ(k+1)}(\psi_{-2}(\boldsymbol{\varepsilon})) = -2$ and $f_{-2}^{\circ n}(\psi_{-2}(\boldsymbol{\varepsilon})) = 2$ for all $n \geq k+2$, which yields $\epsilon_{k+1} = \epsilon_{k+1}' = -1$ and $\epsilon_n = \epsilon_n' = 1$ for all $n \geq k+2$. Thus, the sign sequences $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$ have the desired form.

Conversely, observe that, for $\boldsymbol{\varepsilon} \in \boldsymbol{\Sigma}^{2,1}$ with $\epsilon_1 = -1$ and $\epsilon_2 = 1$, we have

$$\psi_{-2}(\boldsymbol{\varepsilon}) = \epsilon_0 \sqrt{-\beta_{-2} - (-2)} = 0\,.$$

Therefore, if $k \geq 2$ and $\boldsymbol{\varepsilon} \in \boldsymbol{\Sigma}^{k,1}$ satisfies $\epsilon_{k-1} = -1$ and $\epsilon_k = 1$, then

$$\psi_{-2}(\boldsymbol{\varepsilon}) = g_{-2}^{(\epsilon_0, \ldots, \epsilon_{k-3})}\left(\psi_{-2}\left(\sigma^{\circ(k-2)}(\boldsymbol{\varepsilon})\right)\right) = g_{-2}^{(\epsilon_0, \ldots, \epsilon_{k-3})}(0)$$

does not depend on ϵ_{k-2}. This completes the proof of the proposition. □

Remark 19. It follows from Proposition 16 and Proposition 18 that, for all $k \geq 0$ and $p \geq 1$, the set $\mathcal{X}_{-2}^{k,p}$ contains exactly 2^p elements if $k = 0$ and $2^{k+p} - 2^{k-1} + 1$ elements if $k \geq 1$.

Remark 20. Note that we can actually describe the map $\psi_{-2}\colon \mathbf{\Sigma} \to \mathbb{R}$ explicitly. For $\boldsymbol{\varepsilon} \in \mathbf{\Sigma}$, define the sequence $(\delta_n(\boldsymbol{\varepsilon}))_{n\geq 0} \in \{0,1\}^{\mathbb{Z}_{\geq 0}}$ by

$$\delta_n(\boldsymbol{\varepsilon}) = \begin{cases} \delta_{n-1}(\boldsymbol{\varepsilon}) & \text{if } \epsilon_n = 1 \\ 1 - \delta_{n-1}(\boldsymbol{\varepsilon}) & \text{if } \epsilon_n = -1 \end{cases},$$

where $\delta_{-1}(\boldsymbol{\varepsilon}) = 0$ by convention. Then the map $\psi_{-2}\colon \mathbf{\Sigma} \to \mathbb{R}$ is given by

$$\psi_{-2}(\boldsymbol{\varepsilon}) = 2\cos\left(\pi \sum_{n=0}^{+\infty} \frac{\delta_n(\boldsymbol{\varepsilon})}{2^{n+1}}\right).$$

3. Back to the parameter space

We shall now exploit the statements given in Section 2 to get results concerning the parameter space.

Remark 21. By definition, for every point $a \in \mathbb{C}$ and every parameter $c \in \mathbb{C}$, $c \in \mathcal{S}_a$ if and only if $a \in \mathcal{X}_c$ and, for all $k \geq 0$ and $p \geq 1$, $c \in \mathcal{S}_a^{k,p}$ if and only if $a \in \mathcal{X}_c^{k,p}$.

Proposition 22. *For every $a \in \mathbb{C}$, we have*

$$\mathcal{S}_a \subset \{c \in \mathbb{C} : |c| \leq R_a\},$$

where $R_a = |a|^2 + \sqrt{|a|^2+1} + 1$.

Proof. Suppose that $c \in \mathcal{S}_a$. Then, by Proposition 9, we have

$$|c| - |a|^2 \leq |f_c(a)| \leq \rho_c,$$

and hence $\varphi(|c|) \leq |a|^2$, where $\varphi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ is given by

$$\varphi(x) = x - \frac{1+\sqrt{1+4x}}{2}.$$

The map φ is strictly increasing and satisfies $\varphi(R_a) = |a|^2$. Thus, the proposition is proved. □

Now, let us give a more extensive description of $\mathcal{S}_a$ when $a \in (-\infty,-2] \cup [2,+\infty)$.

Given $\epsilon = \pm 1$, let $\mathbf{\Sigma}_\epsilon^{k,p}$ – with $k \geq 0$ and $p \geq 1$ – be the set defined by

$$\mathbf{\Sigma}_\epsilon^{k,p} = \left\{\boldsymbol{\varepsilon} = (\epsilon_n)_{n\geq 0} \in \mathbf{\Sigma}^{k,p} : \epsilon_0 = \epsilon\right\},$$

and let $\mathbf{\Sigma}_\epsilon$ be the set defined by

$$\mathbf{\Sigma}_\epsilon = \bigcup_{k\geq 0,\, p\geq 1} \mathbf{\Sigma}_\epsilon^{k,p} = \{\boldsymbol{\varepsilon} \in \mathbf{\Sigma} : \epsilon_0 = \epsilon\}.$$

For all $k \geq 0$ and $p \geq 1$, the set $\mathbf{\Sigma}_\epsilon^{k,p}$ contains exactly 2^{k+p-1} elements – each of them being completely determined by the choice of its terms with index in $\{1,\dots,k+p-1\}$.

Suppose that $a \in (-\infty,-2] \cup [2,+\infty)$. Then

- for $\boldsymbol{\varepsilon} \in \mathbf{\Sigma}_{\operatorname{sgn}(a)}^{2,1}$ given by $\epsilon_1 = -1$ and $\epsilon_2 = 1$, the map

$$\operatorname{sgn}(a)\zeta_{\boldsymbol{\varepsilon}}\colon c \mapsto \sqrt{-\beta_c - c}$$

  is strictly decreasing on $(-\infty,-2]$ and we have $\zeta_{\boldsymbol{\varepsilon}}(c_a^-) = a$, where c_a^- is the parameter defined by

$$c_a^- = -a^2 - \sqrt{a^2+1} - 1 \in \mathcal{S}_a^{2,1};$$

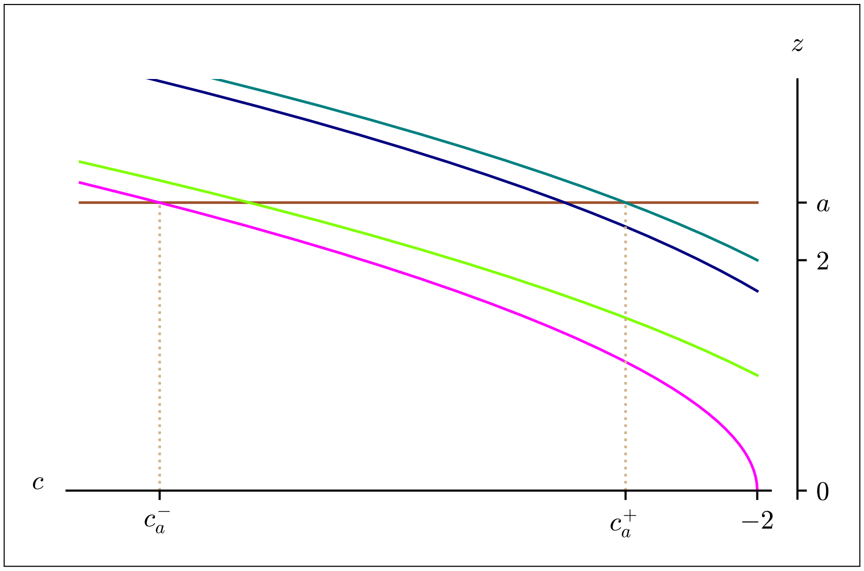

FIGURE 5. Graphs of the maps $\zeta_{\boldsymbol{\varepsilon}}$, with $\boldsymbol{\varepsilon} \in \boldsymbol{\Sigma}_{\operatorname{sgn}(a)}^{2,1}$, when $a \in [2, +\infty)$.

- for $\boldsymbol{\varepsilon} \in \boldsymbol{\Sigma}_{\operatorname{sgn}(a)}^{1,1}$ given by $\epsilon_1 = 1$, the map
$$\operatorname{sgn}(a)\zeta_{\boldsymbol{\varepsilon}} \colon c \mapsto \beta_c$$
is strictly decreasing on $(-\infty, -2]$ and we have $\zeta_{\boldsymbol{\varepsilon}}\left(c_a^+\right) = a$, where c_a^+ is the parameter defined by
$$c_a^+ = -a^2 + |a| \in \mathcal{S}_a^{1,1}\,.$$

Remark 23. Note that, for every $a \in \mathbb{C}$ with $|a| \geq 2$, we have $R_a = -c_{|a|}^-$.

Theorem 24. *Assume that $a \in (-\infty, -2] \cup [2, +\infty)$. Then there is a unique map*
$$\gamma_a \colon \boldsymbol{\Sigma}_{\operatorname{sgn}(a)} \to (-\infty, -2]$$
that satisfies $\zeta_{\boldsymbol{\varepsilon}}\left(\gamma_a(\boldsymbol{\varepsilon})\right) = a$ for all $\boldsymbol{\varepsilon} \in \boldsymbol{\Sigma}_{\operatorname{sgn}(a)}$ (see Figure 5).

Furthermore, we have
$$\mathcal{S}_a^{k,p} = \gamma_a\left(\boldsymbol{\Sigma}_{\operatorname{sgn}(a)}^{k,p}\right) \subset \left[c_a^-, c_a^+\right]\,,$$
for all $k \geq 0$ and $p \geq 1$, (see Figure 6) and the map γ_a is injective.

Claim 25. If $a \in (-\infty, -2] \cup [2, +\infty)$ and $\gamma \in (-\infty, -2]$, then a has at most one preimage under ψ_γ.

Proof of Claim 25. If $\gamma \in (-\infty, -2)$, then the map ψ_γ is injective.

If $\gamma = -2$ and $\boldsymbol{\varepsilon} \in \boldsymbol{\Sigma}$ satisfies $\psi_\gamma(\boldsymbol{\varepsilon}) = a$, then we have
$$2 \leq |a| = |\psi_{-2}(\boldsymbol{\varepsilon})| \leq \beta_{-2} = 2\,,$$
so $\psi_{-2}(\boldsymbol{\varepsilon}) = \operatorname{sgn}(a)\beta_{-2}$, and, by Proposition 18, it follows that $\boldsymbol{\varepsilon}$ is the sign sequence in $\boldsymbol{\Sigma}_{\operatorname{sgn}(a)}^{1,1}$ given by $\epsilon_1 = 1$. Thus, the claim is proved. $\square$

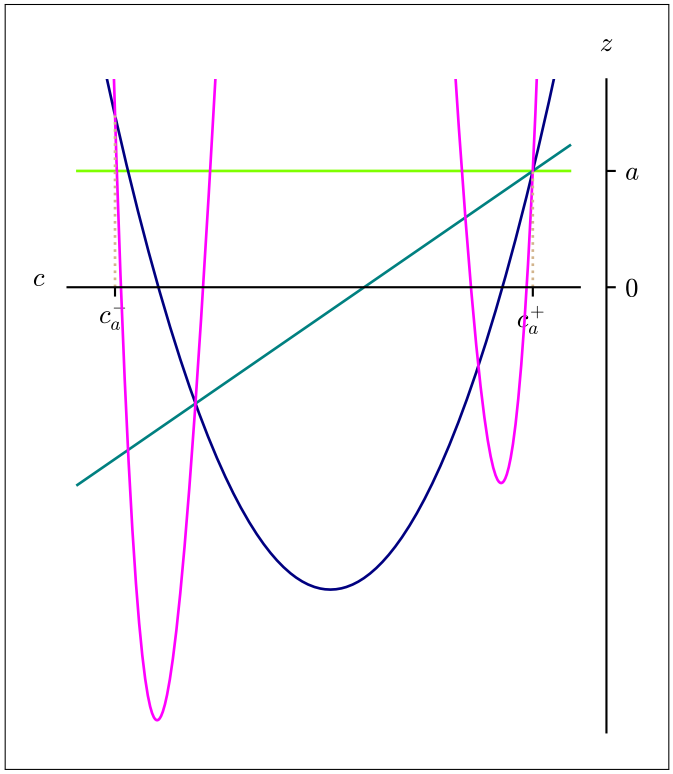

FIGURE 6. Graphs of the maps $c \mapsto F_n(c,a)$, with $n \in \{0, \dots, 3\}$, when $a \in [2, +\infty)$.

Proof of Theorem 24. For every $\boldsymbol{\varepsilon} \in \boldsymbol{\Sigma}_{\mathrm{sgn}(a)}$, we have

$$\mathrm{sgn}(a)\zeta_{\boldsymbol{\varepsilon}}\left(c_a^-\right) \geq \sqrt{-\beta_{c_a^-} - c_a^-} = |a| \quad \text{and} \quad \mathrm{sgn}(a)\zeta_{\boldsymbol{\varepsilon}}\left(c_a^+\right) \leq \beta_{c_a^+} = |a|\,,$$

and hence, by the intermediate value theorem, there exists $\gamma_a(\boldsymbol{\varepsilon}) \in [c_a^-, c_a^+]$ such that $\zeta_{\boldsymbol{\varepsilon}}(\gamma_a(\boldsymbol{\varepsilon})) = a$. Now, note that, if $\boldsymbol{\varepsilon} \in \boldsymbol{\Sigma}_{\mathrm{sgn}(a)}^{k,p}$ – with $k \geq 0$ and $p \geq 1$ – and $\gamma \in (-\infty, -2]$ satisfy $\zeta_{\boldsymbol{\varepsilon}}(\gamma) = a$, then $\boldsymbol{\varepsilon}$ is a preimage of a under ψ_γ, and in particular $\gamma \in \mathcal{S}_a^{k,p}$. Therefore, by Claim 25, the map γ_a so defined is injective, and, as $\mathcal{S}_a^{k,p}$ contains at most 2^{k+p-1} elements, it follows that $\gamma_a\left(\boldsymbol{\Sigma}_{\mathrm{sgn}(a)}^{k,p}\right) = \mathcal{S}_a^{k,p}$, for all $k \geq 0$ and $p \geq 1$. Thus, for every $\boldsymbol{\varepsilon} \in \boldsymbol{\Sigma}_{\mathrm{sgn}(a)}$, $\gamma_a(\boldsymbol{\varepsilon})$ is the unique parameter $\gamma \in (-\infty, -2]$ that satisfies $\zeta_{\boldsymbol{\varepsilon}}(\gamma) = a$. This completes the proof of the theorem. $\square$

Remark 26. Applying Montel's theorem, it follows from Theorem 24 that, for every $a \in (-\infty, -2] \cup [2, +\infty)$, the set of parameters $c \in \mathbb{C}$ for which the point a has bounded forward orbit under f_c is also contained in the line segment $[c_a^-, c_a^+]$.

Note that, when a is an integer, the set $\mathcal{S}_a$ has the following arithmetic property:

Proposition 27. *For every $a \in \mathbb{Z}$, the set $\mathcal{S}_a$ is contained in the set of algebraic integers and is invariant under the action of* $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$.

Proof. For all $k \geq 0$ and $p \geq 1$, the polynomial $F_{k+p}(c,a) - F_k(c,a)$ is monic with integer coefficients since $a \in \mathbb{Z}$. Thus, the proposition is proved. $\square$

We shall now prove Theorem 7, which we recall below.

Theorem 7. *Assume that a and b are two integers with $|b| > |a|$. Then*
- *either $a = 0$, $|b| = 1$ and $\mathcal{S}_a \cap \mathcal{S}_b = \{-2, -1, 0\}$,*
- *or $a = 0$, $|b| = 2$ and $\mathcal{S}_a \cap \mathcal{S}_b = \{-2\}$,*
- *or $|a| \geq 1$, $|b| = |a| + 1$ and $\mathcal{S}_a \cap \mathcal{S}_b = \left\{-a^2 - |a| - 1, -a^2 - |a|\right\}$,*
- *or $|b| > \max\{2, |a| + 1\}$ and $\mathcal{S}_a \cap \mathcal{S}_b = \varnothing$.*

Lemma 28. *Assume that $m \in \mathbb{Z}$ and c is an algebraic integer whose all Galois conjugates lie in the interval $(m - 2, m]$. Then $c = m - 1$ or $c = m$.*

Proof of Lemma 28. Set $\alpha = c - m + 1$. Then α is an algebraic integer whose all Galois conjugates $\alpha_1, \ldots, \alpha_d$ lie in the interval $(-1, 1]$. Therefore, we have

$$\prod_{j=1}^{d} \alpha_j \in (-1, 1] \cap \mathbb{Z} = \{0, 1\} ,$$

and it follows that either $\alpha_j = 0$ for some $j \in \{1, \ldots, d\}$, which yields $\alpha = 0$, or $\alpha_j = 1$ for all $j \in \{1, \ldots, d\}$. Thus, either $c = m - 1$ or $c = m$. $\square$

Proof of Theorem 7. For a proof of the case $a = 0$ and $|b| = 1$, we refer the reader to [Buf18, Proposition 6].

Thus, we may assume that $|b| \geq 2$. By Proposition 22, Theorem 24 and Proposition 27, the set $\mathcal{S}_a \cap \mathcal{S}_b$ is contained in the set of algebraic integers, is invariant under the action of $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ and satisfies

$$\mathcal{S}_a \cap \mathcal{S}_b \subset \{c \in \mathbb{C} : |c| \leq R_a\} \cap \left[c_b^-, c_b^+\right] .$$

Suppose that $a = 0$. Then we have

$$c_b^+ = -b^2 + |b| \leq -2 = -R_a ,$$

with equality if and only if $|b| = 2$. Therefore, $\mathcal{S}_a \cap \mathcal{S}_b \subset \{-2\}$ if $|b| = 2$ and $\mathcal{S}_a \cap \mathcal{S}_b = \varnothing$ otherwise. Conversely, observe that $-2 \in \mathcal{S}_a^{2,1} \cap \mathcal{S}_b^{1,1}$ when $|b| = 2$.

Now, suppose that $|a| \geq 1$. Then we have

$$c_b^+ - 2 < -R_a = -a^2 - \sqrt{a^2 + 1} - 1 < -a^2 - |a| = c_b^+ \quad \text{if} \quad |b| = |a| + 1$$

and

$$c_b^+ = -b^2 + |b| < -a^2 - \sqrt{a^2 + 1} - 1 = -R_a \quad \text{if} \quad |b| \geq |a| + 2 .$$

Therefore, $\mathcal{S}_a \cap \mathcal{S}_b \subset \left\{-a^2 - |a| - 1, -a^2 - |a|\right\}$ if $|b| = |a| + 1$ by Lemma 28 and $\mathcal{S}_a \cap \mathcal{S}_b = \varnothing$ otherwise. Conversely, observe that $-a^2 - |a| - 1 \in \mathcal{S}_a^{1,2} \cap \mathcal{S}_b^{1,2}$ and $-a^2 - |a| \in \mathcal{S}_a^{1,1} \cap \mathcal{S}_b^{1,1}$ when $|b| = |a| + 1$. Thus, the theorem is proved. $\square$

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