MODULI SPACES OF POLYNOMIAL MAPS AND MULTIPLIERS AT SMALL CYCLES

VALENTIN HUGUIN

Abstract. Fix an integer $d \geq 2$. The space $\mathcal{P}_d$ of polynomial maps of degree d modulo conjugation by affine transformations is naturally an affine variety over $\mathbb{Q}$ of dimension $d-1$. For each integer $P \geq 1$, the elementary symmetric functions of the multipliers at all the cycles with period $p \in \{1, \dots, P\}$ induce a natural morphism $\operatorname{Mult}_d^{(P)}$ defined on $\mathcal{P}_d$. In this article, we show that the morphism $\operatorname{Mult}_d^{(2)}$ induced by the multipliers at the cycles with periods 1 and 2 is both finite and birational onto its image. In the case of polynomial maps, this strengthens results by McMullen and by Ji and Xie stating that $\operatorname{Mult}_d^{(P)}$ is quasifinite and birational onto its image for all sufficiently large integers P. Our result arises as the combination of the following two statements:

- A sequence of polynomials over $\mathbb{C}$ of degree d with bounded multipliers at its cycles with periods 1 and 2 is necessarily bounded in $\mathcal{P}_d(\mathbb{C})$.
- A generic conjugacy class of polynomials over $\mathbb{C}$ of degree d is uniquely determined by its multipliers at its cycles with periods 1 and 2.

1. Introduction

Fix any integer $d \geq 2$. We wish here to describe the space Poly_d of polynomials of degree d from a dynamical perspective. The group Aff of affine transformations acts on Poly_d by conjugation, via $\phi \centerdot f = \phi \circ f \circ \phi^{-1}$. Since conjugate polynomials induce the same dynamical system, up to some change of coordinates, it is natural to consider the space $\mathcal{P}_d = \operatorname{Poly}_d / \operatorname{Aff}$ of conjugacy classes of polynomial maps of degree d. This quotient space $\mathcal{P}_d$ is naturally an affine variety over $\mathbb{Q}$ of dimension $d-1$ called the *moduli space of polynomial maps* of degree d. As an affine variety, $\mathcal{P}_d$ is completely determined by its coordinate ring $\mathbb{Q}[\mathcal{P}_d]$, consisting of all regular functions defined on $\mathcal{P}_d$. From a dynamical point of view, natural regular functions on $\mathcal{P}_d$ are given by the elementary symmetric functions of the multipliers at all the cycles with any given period. Thus, it is natural to ask how well the multipliers at the cycles describe the moduli space $\mathcal{P}_d$.

Suppose that K is an arbitrary algebraically closed field of characteristic 0 and $f \in \operatorname{Poly}_d(K)$. Recall that a point $z_0 \in K$ is *periodic* for f if there is some integer $p \geq 1$ such that $f^{\circ p}(z_0) = z_0$. In this case, the smallest such integer p is called the *period* of z_0 and $\{f^{\circ n}(z_0) : n \geq 0\}$ is said to be a *cycle* for f. The *multiplier* of f at z_0 is the number $\lambda = (f^{\circ p})'(z_0) \in K$. The polynomial f has the same multiplier at each point of the cycle. Moreover, the multiplier is invariant under conjugation: if $\phi \in \operatorname{Aff}(K)$, then $\phi(z_0)$ is periodic for $\phi \centerdot f$ with period p and multiplier λ.

2020 *Mathematics Subject Classification.* Primary 37F46, 37P45; Secondary 37F10, 37P05, 37P30.

The research of the author was partly supported by the German Research Foundation (DFG, project number 455038303).

Given $f \in \mathrm{Poly}_d(K)$, with K an algebraically closed field of characteristic 0, for $p \geq 1$, we denote by $\Lambda_f^{(p)} \in K^{N_d^{(p)}}/\mathfrak{S}_{N_d^{(p)}}$ the multiset of multipliers of f at all its cycles with period p, which depends only on the conjugacy class $[f] \in \mathcal{P}_d(K)$. We ask here if $f \in \mathrm{Poly}_d(K)$, with K an algebraically closed field of characteristic 0, is characterized by its multiplier spectrum $\left(\Lambda_f^{(p)}\right)_{p \geq 1}$. Since $\mathcal{P}_d$ is finite-dimensional, we can even ask if $\mathcal{P}_d$ is well described by the multipliers at the cycles with period at most P for some integer $P \geq 1$.

For $n \geq 1$, we denote by $\mathbb{A}^n$ the affine space of dimension n over $\mathbb{Q}$. For $N \geq 1$ and every algebraically closed field K, we have a natural bijection $K^N/\mathfrak{S}_N \cong K^N$ given by the elementary symmetric functions. Via these identifications, for $P \geq 1$, we consider the *multiplier spectrum morphism* $\mathrm{Mult}_d^{(P)} \colon \mathcal{P}_d \to \prod_{p=1}^{P} \mathbb{A}^{N_d^{(p)}}$ given by

$$\mathrm{Mult}_d^{(P)}([f]) = \left(\Lambda_f^{(1)}, \dots, \Lambda_f^{(P)}\right) .$$

Here, we show that the multipliers at the cycles with periods 1 and 2 provide a good description of the space $\mathcal{P}_d$. More precisely, our main result is the following:

Main Theorem. *Assume that $d \geq 2$ is an integer. Then $\mathrm{Mult}_d^{(2)}$ induces a finite birational morphism from $\mathcal{P}_d$ onto its image $\Sigma_d^{(2)}$. Furthermore, if $d \in \{2, 3\}$, then $\mathrm{Mult}_d^{(1)}$ induces an isomorphism from $\mathcal{P}_d$ onto its image $\Sigma_d^{(1)}$.*

Remark 1. It follows from Main Theorem that $\mathrm{Mult}_d^{(P)}$ induces a finite birational morphism from $\mathcal{P}_d$ onto its image $\Sigma_d^{(P)}$ for all $P \geq 2$. Moreover, if $d \in \{2, 3\}$, then $\mathrm{Mult}_d^{(P)}$ induces an isomorphism from $\mathcal{P}_d$ onto its image $\Sigma_d^{(P)}$ for all $P \geq 1$.

Remark 2. In general, $\mathrm{Mult}_d^{(P)}$ may not be an isomorphism onto its image $\Sigma_d^{(P)}$ for any $P \geq 1$ (see Appendix B for details).

In fact, Main Theorem is a direct combination of Theorems A and C, which we now present.

1.1. Degeneration of complex polynomial maps and multipliers at small cycles.

It is natural to investigate the behavior of multipliers under degeneration in the space $\mathcal{P}_d(\mathbb{C})$ of affine conjugacy classes of complex polynomials of degree d. This study was notably conducted by DeMarco and McMullen in [DM08]. We ask here if degeneration in the space $\mathcal{P}_d(\mathbb{C})$ is always detected by the multipliers at the cycles with small periods.

The set $\mathcal{P}_d(\mathbb{C})$ of conjugacy classes of complex polynomial maps of degree d is naturally a complex orbifold of dimension $d-1$. We say that a sequence $(f_n)_{n \geq 0}$ of elements of $\mathrm{Poly}_d(\mathbb{C})$ *degenerates* in $\mathcal{P}_d(\mathbb{C})$ if the sequence $([f_n])_{n \geq 0}$ eventually leaves every compact subset of $\mathcal{P}_d(\mathbb{C})$. We can also express degeneration in $\mathcal{P}_d(\mathbb{C})$ in terms of maximal escape rates.

Given an algebraically closed field K of characteristic 0 equipped with an absolute value $|.|$ and $f \in \mathrm{Poly}_d(K)$, the *Green function* $g_f \colon K \to \mathbb{R}_{\geq 0}$ is given by

$$g_f(z) = \lim_{n \to +\infty} \frac{1}{d^n} \log^+ \left|f^{\circ n}(z)\right|$$

and the *maximal escape rate* M_f of f is defined by

$$M_f = \max \left\{g_f(c) : c \in K,\ f'(c) = 0\right\} .$$

The maximal escape rate is invariant under conjugation. Moreover, in the complex case, the maximal escape rate characterizes degeneration in $\mathcal{P}_d(\mathbb{C})$. More precisely, any sequence $(f_n)_{n\geq 0}$ of elements of $\mathrm{Poly}_d(\mathbb{C})$ degenerates in $\mathcal{P}_d(\mathbb{C})$ if and only if $\lim_{n\to+\infty} M_{f_n} = +\infty$.

Now, given an algebraically closed field K of characteristic 0 equipped with an absolute value $|.|$ and $f \in \mathrm{Poly}_d(K)$, for $p \geq 1$, we define

$$M_f^{(p)} = \max_{\lambda \in \Lambda_f^{(p)}} \left(\frac{1}{p} \log|\lambda| \right)$$

to be the maximal characteristic exponent of f at a cycle with period p.

By [Oku12], for every algebraically closed valued field K of characteristic 0 and every $f \in \mathrm{Poly}_d(K)$, we have

$$\lim_{p\to+\infty} \frac{1}{N_d^{(p)}} \sum_{\lambda \in \Lambda_f^{(p)}} \left(\frac{1}{p} \log|\lambda| \right) = \log|d| + \sum_{c \in \Gamma_f} \rho_c \cdot g_f(c)\,,$$

where $\Gamma_f \subseteq K$ is the set of critical points for f and $\rho_c \geq 1$ denotes the multiplicity of c as a critical point for f for each $c \in \Gamma_f$, which yields $\sup_{p\geq 1} M_f^{(p)} \geq M_f + \log|d|$. In particular, if $(f_n)_{n\geq 0}$ is any sequence of elements of $\mathrm{Poly}_d(\mathbb{C})$ that degenerates in $\mathcal{P}_d(\mathbb{C})$, then $\lim_{n\to+\infty} \sup_{p\geq 1} M_{f_n}^{(p)} = +\infty$. Thus, degeneration in $\mathcal{P}_d(\mathbb{C})$ is detected by the full multiplier spectrum.

We show here that degeneration in $\mathcal{P}_d(\mathbb{C})$ is already detected by the multipliers at the cycles with periods 1 and 2. Explicitly, we obtain the following:

Theorem A. *Assume that $d \geq 2$ is an integer and $(f_n)_{n\geq 0}$ is any sequence of elements of $\mathrm{Poly}_d(\mathbb{C})$ that degenerates in $\mathcal{P}_d(\mathbb{C})$. Then*

$$\lim_{n\to+\infty} \max\left\{ M_{f_n}^{(1)}, M_{f_n}^{(2)} \right\} = +\infty\,.$$

Furthermore, if $d \in \{2,3\}$, then $\lim_{n\to+\infty} M_{f_n}^{(1)} = +\infty$.

As a consequence of Theorem A, the morphism $\mathrm{Mult}_d^{(2)}$ given by the multipliers at the cycles with periods 1 and 2 is proper, and hence finite since $\mathcal{P}_d$ is an affine variety. In the polynomial case, this strengthens a result established by McMullen in [McM87] stating that $\mathrm{Mult}_d^{(P)}$ is a quasifinite morphism for some integer $P \geq 1$. By contrast, Fujimura proved in [Fuj07] that the morphism $\mathrm{Mult}_d^{(1)}$ induced by the multipliers at the fixed points is neither quasifinite nor surjective onto its scheme-theoretic image $\Sigma_d^{(1)}$ if $d \geq 4$.

Then, using the fact that $\mathrm{Mult}_d^{(2)}$ is a finite morphism, we generalize Theorem A to polynomials over any algebraically closed valued field of characteristic 0. More precisely, we obtain the following stronger result:

Corollary A.1. *Suppose that $d \geq 2$ is an integer and K is an algebraically closed valued field of characteristic 0. Then there exist $A \in \mathbb{R}_{>0}$ and $B \in \mathbb{R}$ such that*

$$\max\left\{ M_f^{(1)}, M_f^{(2)} \right\} \geq A \cdot M_f + B$$

for all $f \in \mathrm{Poly}_d(K)$. Moreover, if $d \in \{2,3\}$, then there exist $A \in \mathbb{R}_{>0}$ and $B \in \mathbb{R}$ such that $M_f^{(1)} \geq A \cdot M_f + B$ for all $f \in \mathrm{Poly}_d(K)$.

We also deduce a relation between the critical height of any polynomial defined over a number field and the standard heights of its multipliers at the small cycles. Here, denote by $h\colon \overline{\mathbb{Q}} \to \mathbb{R}_{\geq 0}$ the standard height on the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. Given $f \in \mathrm{Poly}_d\left(\overline{\mathbb{Q}}\right)$, define H_f to be the critical height of f and, for $p \geq 1$, define $H_f^{(p)} = \max\limits_{\lambda \in \Lambda_f^{(p)}} h(\lambda)$ (see Subsection 3.5 for details). We obtain the following:

Corollary A.2. *Suppose that $d \geq 2$ is an integer. Then there exist $A \in \mathbb{R}_{>0}$ and $B \in \mathbb{R}$ such that*

$$\max\left\{H_f^{(1)}, H_f^{(2)}\right\} \geq A \cdot H_f + B$$

for all $f \in \mathrm{Poly}_d\left(\overline{\mathbb{Q}}\right)$. Moreover, if $d \in \{2, 3\}$, then there exist $A \in \mathbb{R}_{>0}$ and $B \in \mathbb{R}$ such that $H_f^{(1)} \geq A \cdot H_f + B$ for all $f \in \mathrm{Poly}_d\left(\overline{\mathbb{Q}}\right)$.

In fact, we obtain Theorem A as a direct consequence of the quantitative result below. In most cases, this is a stronger version of Corollary A.1. Furthermore, the bounds in our statement are optimal.

Theorem B. *Assume that $d \geq 2$ is an integer, K is an algebraically closed valued field of characteristic 0 that is either Archimedean or non-Archimedean with residue characteristic 0 or greater than d and $f \in \mathrm{Poly}_d(K)$. If $d \geq 4$, then*

$$M_f^{(1)} \geq \frac{d-1}{d-2} M_f \quad \text{or} \quad M_f^{(2)} \geq C_d \cdot M_f\,, \quad \text{with} \quad C_d = \begin{cases} \frac{2(d-1)}{d} & \text{if } d \text{ is even} \\ \frac{2d}{d+1} & \text{if } d \text{ is odd} \end{cases} .$$

In addition, $M_f^{(1)} \geq M_f$ if $d = 2$, and $M_f^{(1)} \geq 2M_f$ if $d = 3$.

Remark 3. Alternatively, one can establish Theorem A by relating bounded multipliers to rescalings for sequences of polynomial maps and by counting some critical points. Suppose that $(f_n)_{n\geq 0}$ is a sequence of elements of $\mathrm{Poly}_d(\mathbb{C})$. We say that a sequence $(\phi_n)_{n\geq 0}$ of elements of $\mathrm{Aff}(\mathbb{C})$ is a *rescaling* for $(f_n)_{n\geq 0}$ with period $p \geq 1$ and degree $e \geq 2$ if $\left(\phi_n \circ f_n^{\circ p} \circ \phi_n^{-1}\right)_{n\geq 0}$ converges locally uniformly on $\mathbb{C}$ to some $g \in \mathrm{Poly}_e(\mathbb{C})$. We say that two rescalings $(\phi_n)_{n\geq 0}$ and $(\psi_n)_{n\geq 0}$ are *independent* if, for each bounded subset D of $\mathbb{C}$, we have $\phi_n^{-1}(D) \cap \psi_n^{-1}(D) = \varnothing$ for all sufficiently large n. More precisely, to prove Theorem A, one can proceed as follows: Assume that both $\sup\limits_{n\geq 0} M_{f_n}^{(1)} < +\infty$ and $\sup\limits_{n\geq 0} M_{f_n}^{(2)} < +\infty$. Then, according to the discussion in [FT08, Section 2], possibly passing to some subsequence, $(f_n)_{n\geq 0}$ has pairwise independent rescalings $(\phi_{j,n})_{n\geq 0}$ with period 1 and respective degrees $d_j \geq 2$, with $r \geq 1$ and $j \in \{1, \ldots, r\}$, such that $\sum\limits_{j=1}^{r} d_j = d$. If $d \in \{2, 3\}$, then $r = 1$, and hence $(f_n)_{n\geq 0}$ does not degenerate in $\mathcal{P}_d(\mathbb{C})$. Thus, assume that $d \geq 4$. Using again the arguments of [FT08, Section 2] and passing to a subsequence if necessary, $(f_n)_{n\geq 0}$ has pairwise independent rescalings $(\psi_{k,n})_{n\geq 0}$ with period 2 and respective degrees $e_k \geq 2$, with $s \geq 0$ and $k \in \{1, \ldots, s\}$, that are independent from all the $(\phi_{j,n})_{n\geq 0}$, with $j \in \{1, \ldots, r\}$, and such that $\sum\limits_{j=1}^{r} d_j^2 + \sum\limits_{k=1}^{s} e_k = d^2$. Finally, given a sufficiently large bounded subset D of $\mathbb{C}$, define $N \geq 0$ to be the number of critical points for $f_n^{\circ 2}$ in $\bigcup\limits_{k=1}^{s} \psi_{k,n}^{-1}(D)$ for all sufficiently large n, counting multiplicities. Then one can show that $N \geq 2(d-1)(r-1)$ and $N \leq (d+1)(r-1)$, which yields $r = 1$. Thus, the sequence $(f_n)_{n \geq 0}$ does not degenerate in $\mathcal{P}_d(\mathbb{C})$. We refer the reader to [Fav24] for a concise and complete proof of Theorem A using analogous arguments, which considers the dynamical system induced by a meromorphic family of rational maps on a certain Berkovich projective line (see [Kiw15] for the correspondence between rescalings and periodic points of type II in this Berkovich space).

Remark 4. While we provide lower bounds on $\max\left\{M_f^{(1)}, M_f^{(2)}\right\}$ in terms of M_f, with K an algebraically closed valued field of characteristic 0 and $f \in \mathrm{Poly}_d(K)$, it is not difficult to prove reverse inequalities. In fact, for every $p \geq 1$, one can easily establish an upper bound on $M_f^{(p)}$ in terms of M_f, with K an algebraically closed valued field of characteristic 0 and $f \in \mathrm{Poly}_d(K)$ (see Appendix A for details).

Remark 5. Actually, one can prove that, if $(f_n)_{n \geq 0}$ is any sequence of elements of $\mathrm{Poly}_d(\mathbb{C})$ that degenerates in $\mathcal{P}_d(\mathbb{C})$, then $\lim\limits_{n \to +\infty} M_{f_n}^{(2)} = +\infty$. Furthermore, under the hypotheses of Theorem B, one can show that $M_f^{(2)} \geq M_f$ if $M_f > 0$.

1.2. **Determination of generic conjugacy classes of polynomial maps by their multipliers at their small cycles.** It is natural to ask how many polynomial maps have the same multipliers, up to conjugation.

Fujimura showed in [Fuj07] that the induced morphism $\mathrm{Mult}_d^{(1)} \colon \mathcal{P}_d \to \Sigma_d^{(1)}$ has degree $(d-2)!$ (see also [Sug17] and [Sug23]). Thus, generically, there are exactly $(d-2)!$ elements of $\mathcal{P}_d(\mathbb{C})$ that have the same multiset of multipliers at their fixed points. Here, we prove that a generic element of $\mathcal{P}_d(\mathbb{C})$ is uniquely determined by its multipliers at its cycles with periods 1 and 2.

Theorem C. *Assume that $d \geq 2$ is an integer. Then there is a nonempty Zariski-open subset U of $\mathcal{P}_d$ such that each $[f] \in U(\mathbb{C})$ is the unique $[g] \in \mathcal{P}_d(\mathbb{C})$ such that $\Lambda_g^{(1)} = \Lambda_f^{(1)}$ and $\Lambda_g^{(2)} = \Lambda_f^{(2)}$. Moreover, if $d \in \{2, 3\}$, then each $[f] \in \mathcal{P}_d(\mathbb{C})$ is the unique $[g] \in \mathcal{P}_d(\mathbb{C})$ such that $\Lambda_g^{(1)} = \Lambda_f^{(1)}$.*

In other words, Theorem C states that the morphism $\mathrm{Mult}_d^{(2)}$ is birational onto its image $\Sigma_d^{(2)}$. This proves a conjecture made by Hutz and Tepper in [HT13], who had checked it when $d \in \{2, 3, 4, 5\}$. This also strengthens a recent result obtained by Ji and Xie in [JX24] asserting that $\mathrm{Mult}_d^{(P)}$ is birational onto its image $\Sigma_d^{(P)}$ for some integer $P \geq 1$.

Remark 6. In general, there may exist distinct elements $[f], [g] \in \mathcal{P}_d(\mathbb{C})$ such that $\Lambda_f^{(1)} = \Lambda_g^{(1)}$ and $\Lambda_f^{(2)} = \Lambda_g^{(2)}$. For $d = 4$, we can describe precisely when this occurs (see Appendix B).

1.3. **Known results in the case of rational maps.** The algebraic group PSL_2 of Möbius transformations acts on the space Rat_d of rational maps of degree d by conjugation. The quotient $\mathcal{M}_d$ forms an affine variety over $\mathbb{Q}$ of dimension $2d-2$ called the *moduli space of rational maps* of degree d. As in the polynomial setting, for each $P \geq 1$, the elementary symmetric functions of the multipliers at the cycles with period $p \in \{1, \dots, P\}$ define a morphism $\widehat{\mathrm{Mult}}_d^{(P)} \colon \mathcal{M}_d \to \prod_{p=1}^{P} \mathbb{A}^{\widehat{N}_d^{(p)}}$.

Milnor showed in [Mil93] that the morphism $\widehat{\mathrm{Mult}}_2^{(1)}$ given by the multipliers of quadratic rational maps at the fixed points is an isomorphism onto its image $\widehat{\Sigma}_2^{(1)}$.

Unlike the case of polynomial maps, if $d \geq 4$, then the morphism $\widehat{\mathrm{Mult}}_d^{(P)}$ is not proper for any $P \geq 1$. For example, if d is a perfect square, taking flexible Lattès maps, one can find degenerating sequences in $\mathcal{M}_d(\mathbb{C})$ whose elements all have the same multisets of multipliers for each period. As another example, first examined by McMullen in [McM88], the sequence $(f_n)_{n\geq 1}$ of elements of $\mathrm{Rat}_5(\mathbb{C})$ defined by $f_n(z) = z^2 + \frac{1}{nz^3}$ degenerates in $\mathcal{M}_5(\mathbb{C})$ and has uniformly bounded multipliers for each period. We refer to [Luo22] for a description of the hyperbolic components of $\mathcal{M}_d(\mathbb{C})$ containing degenerating sequences with uniformly bounded multipliers for each period. We also refer to [Fav24] and [Gon24] for very recent results about the behavior of multipliers under degeneration in $\mathcal{M}_d(\mathbb{C})$.

Now, denote by $\mathcal{L}_d \subseteq \mathcal{M}_d$ the locus of conjugacy classes of flexible Lattès maps of degree d. Note that $\mathcal{L}_d$ is empty if d is not a perfect square, $\mathcal{L}_d$ is an irreducible curve if d is an even square and $\mathcal{L}_d$ is the union of two irreducible curves if d is an odd square (see [Mil06]). McMullen showed that, aside from flexible Lattès maps, any conjugacy class of rational maps of degree d is determined up to finitely many choices by its multiplier spectrum. Thus, McMullen established the following:

Theorem 7 ([McM87, Corollary 2.3])**.** *There exists an integer $P \geq 1$ such that the restriction of $\widehat{\mathrm{Mult}}_d^{(P)}$ to $\mathcal{M}_d \setminus \mathcal{L}_d$ is a quasifinite morphism.*

In [GOV20], Gauthier, Okuyama and Vigny exhibited some integer $P \geq 1$ as in Theorem 7 that can be explicitly computed from quantities related to bifurcations. In addition, Ji and Xie established in [JX23] that, aside from flexible Lattès maps, any conjugacy class of rational maps of degree d is already determined up to only finitely many choices by its multisets of moduli of multipliers for each period.

Very recently, Ji and Xie also showed that a generic conjugacy class of rational maps of degree d is uniquely determined by its multiplier spectrum.

Theorem 8 ([JX24, Theorem 1.3])**.** *There exists an integer $P \geq 1$ such that the morphism $\widehat{\mathrm{Mult}}_d^{(P)}$ is birational onto its scheme-theoretic image $\widehat{\Sigma}_d^{(P)}$.*

Moreover, Ji and Xie conjectured that Theorem C also holds for rational maps or, equivalently, that the morphism $\widehat{\mathrm{Mult}}_d^{(2)}$ is birational onto its image $\widehat{\Sigma}_d^{(2)}$. This conjecture was proved for cubic rational maps by Gotou in [Got23].

1.4. **Outline of the paper.** For the reader's convenience, the sections are mostly independent from each other.

In Section 2, we show that the moduli space $\mathcal{P}_d$ exists as a geometric quotient, we provide a precise definition of the multiplier spectrum morphisms $\mathrm{Mult}_d^{(P)}$, with $P \geq 1$, and we examine the cases of quadratic and cubic polynomials. In Section 3, we prove Theorem B in the complex setting and we derive Corollaries A.1 and A.2 from Theorem A. In Section 5, we prove Theorem C. In Appendix A, we obtain a few additional estimates on absolute values of multipliers of polynomial maps. In Appendix B, we discuss isospectral polynomial maps and we describe the pairs of quartic polynomials that have the same multipliers for periods 1 and 2.

Acknowledgments. The author would like to thank Xavier Buff for helpful discussions, and in particular for stating a two-islands lemma. The author would like to thank Charles Favre for inquiring about Theorem A, which was the initial motivation for this paper, and for suggesting to investigate the non-Archimedean setting. The author would also like to thank Igors Gorbovickis for helpful discussions.

2. Moduli spaces of polynomial maps and multiplier spectrum morphisms

Throughout this section, we fix an integer $d \geq 2$.

2.1. Moduli spaces of polynomial maps.

First, let us present the space $\mathcal{P}_d$ of polynomial maps of degree d modulo conjugation by an affine transformation, and specifically let us describe its algebraic and complex analytic structures. Here, we shall elaborate on the similar discussion in [FG22, Section 2.1].

We shall start by showing that this space $\mathcal{P}_d$ exists as a geometric quotient and affine variety over $\mathbb{Q}$. The analogous statement about the space $\mathcal{M}_d$ of all rational maps of degree d modulo conjugation by a Möbius transformation was obtained by Silverman in [Sil98] by using the geometric invariant theory developed by Mumford in [MFK94].

As we work over a non-algebraically closed field, we shall first briefly recall some material from algebraic geometry. For more details, we refer the reader to [Mil17], [MFK94] and [Poo17].

Here, we call *variety* over $\mathbb{Q}$ any scheme X that is geometrically reduced, separated and of finite type over $\mathbb{Q}$. For each variety X over $\mathbb{Q}$ and each commutative $\mathbb{Q}$-algebra R, we denote by $X(R)$ the set of R-valued points of X. For each $n \geq 1$, we denote by $\mathbb{A}^n$ the affine space of dimension n over $\mathbb{Q}$.

Here, we call *algebraic group* over $\mathbb{Q}$ any variety G over $\mathbb{Q}$ together with a point $e \in G(\mathbb{Q})$ and morphisms $m\colon G \times G \to G$ and $\mathrm{inv}\colon G \to G$ of varieties over $\mathbb{Q}$ that satisfy the usual group axioms, where e, m and inv represent the identity element, the group law and the inversion, respectively. Given any algebraic group G over $\mathbb{Q}$ and any commutative $\mathbb{Q}$-algebra R, the set $G(R)$ has a natural group structure.

Given an algebraic group G over $\mathbb{Q}$ and a variety X over $\mathbb{Q}$, we call *action* of G on X any morphism $\theta\colon G \times X \to X$ of varieties over $\mathbb{Q}$ such that the induced map $\theta\colon G(R) \times X(R) \to X(R)$ is a group action for each commutative $\mathbb{Q}$-algebra R.

Suppose that G is an algebraic group over $\mathbb{Q}$, X is a variety over $\mathbb{Q}$ and G acts on X via a morphism $\theta\colon G \times X \to X$. For each commutative $\mathbb{Q}$-algebra R, denote here by $(g, x) \mapsto g \centerdot x$ the induced group action $\theta\colon G(R) \times X(R) \to X(R)$. Given a variety Z over $\mathbb{Q}$, we say that a morphism $\Psi\colon X \to Z$ is *invariant* under the action θ if, for every commutative $\mathbb{Q}$-algebra R, we have $\Psi(g \centerdot x) = \Psi(x)$ for all $g \in G(R)$ and all $x \in X(R)$. In fact, given an algebraically closed field K of characteristic 0, any morphism $\Psi\colon X \to Z$, with Z a variety over $\mathbb{Q}$, is invariant under θ if and only if $\Psi(g \centerdot x) = \Psi(x)$ for all $g \in G(K)$ and all $x \in X(K)$. If X is affine, we denote by $\mathbb{Q}[X]^G$ the commutative $\mathbb{Q}$-algebra of all regular functions on X that are invariant under θ when viewed as morphisms from X to $\mathbb{A}^1$.

Suppose again that some algebraic group G over $\mathbb{Q}$ acts on a variety X over $\mathbb{Q}$. Here, we call *geometric quotient* of X by G any variety X/G over $\mathbb{Q}$ together with a morphism $\pi\colon X \to X/G$ that satisfies the following conditions:

(1) for any algebraically closed field K of characteristic 0, we have

$$(X/G)(K) = X(K)/G(K)\,,$$

in the sense that the induced map $\pi\colon X(K) \to (X/G)(K)$ is surjective and its fibers are precisely the orbits $\{g \centerdot x : g \in G(K)\}$, with $x \in X(K)$;

(2) the space X/G has the quotient topology: any subset U of X/G is open if and only if $\pi^{-1}(U)$ is an open subset of X;

(3) for every open subset U of X/G, each regular function ψ on $\pi^{-1}(U)$ that is invariant under the induced action of G on $\pi^{-1}(U)$ factors as $\psi = \overline{\psi} \circ \pi$, with $\overline{\psi}$ a regular function on U.

If a variety X/G over $\mathbb{Q}$ with a morphism $\pi\colon X \to X/G$ is a geometric quotient of X by G, then it is also a *categorical quotient*: each invariant morphism $\Psi\colon X \to Z$, with Z a variety over $\mathbb{Q}$, factors as $\Psi = \overline{\Psi} \circ \pi$ in a unique way, with $\overline{\Psi}\colon X/G \to Z$ a morphism. In particular, a geometric quotient of X by G (if it exists) is unique, up to isomorphism.

We shall now state general results about existence of geometric quotients. First, we have the well-known result below about geometric quotients of affine varieties by finite algebraic groups. We omit here the proof and refer to [SGA70, Exposé V, Théorème 4.1] for a more general statement.

Lemma 9. *Suppose that G is some finite algebraic group over $\mathbb{Q}$ that acts on an affine variety X over $\mathbb{Q}$. Then $\mathbb{Q}[X]^G$ is a finitely generated commutative $\mathbb{Q}$-algebra. Moreover, the affine variety X/G over $\mathbb{Q}$ such that $\mathbb{Q}[X/G] = \mathbb{Q}[X]^G$ together with the morphism $\pi\colon X \to X/G$ induced by the inclusion $\mathbb{Q}[X]^G \subseteq \mathbb{Q}[X]$ is a geometric quotient of X by G. Furthermore, π is a finite morphism.*

Now, suppose that an algebraic group G over $\mathbb{Q}$ acts on a variety X over $\mathbb{Q}$ and Y is a closed subvariety of X. We call *stabilizer* of Y the algebraic subgroup H of G such that $H(K) = \{g \in G(K) : g \centerdot Y(K) \subseteq Y(K)\}$, where K is any algebraically closed field of characteristic 0. Note that the action of G on X yields an action of H on Y. If X is affine and H is finite, then a geometric quotient Y/H of Y by H exists by Lemma 9. Under some additional assumption implying that every orbit $\{g \centerdot x : g \in G(K)\}$, with $x \in X(K)$, has a nonempty intersection with $Y(K)$, where K is any algebraically closed field of characteristic 0, a geometric quotient X/G of X by G also exists and $X/G \cong Y/H$. More precisely, we have the following result. We omit the proof and refer to [SGA70, Exposé V, Lemme 6.1] for a more general statement expressed in terms of groupoids.

Lemma 10. *Suppose that G is some algebraic group over $\mathbb{Q}$ that acts on an affine variety X over $\mathbb{Q}$ via a morphism $\theta\colon G \times X \to X$. Also assume that there exists a closed subvariety Y of X such that the induced morphism $\theta\colon G \times Y \to X$ is finite, flat and surjective. Then the stabilizer H of Y is finite and the closed immersion $\imath\colon Y \hookrightarrow X$ induces an isomorphism $\imath^*\colon \mathbb{Q}[X]^G \to \mathbb{Q}[Y]^H$ of $\mathbb{Q}$-algebras. Therefore, a geometric quotient Y/H of Y by H exists, $\mathbb{Q}[X]^G$ is a finitely generated commutative $\mathbb{Q}$-algebra and the affine variety X/G over $\mathbb{Q}$ such that $\mathbb{Q}[X/G] = \mathbb{Q}[X]^G$ is isomorphic to Y/H. Furthermore, X/G together with the morphism $\pi\colon X \to X/G$ induced by the inclusion $\mathbb{Q}[X]^G \subseteq \mathbb{Q}[X]$ is a geometric quotient of X by G.*

We now turn to the construction of the moduli space $\mathcal{P}_d$ of polynomial maps of degree d. Consider the space

$$\mathrm{Poly}_d = \left\{a_d z^d + \cdots + a_1 z + a_0 : a_d \neq 0\right\}$$

of all polynomial maps of degree d. Identifying a polynomial with its coefficients, Poly_d is naturally an affine variety over $\mathbb{Q}$ such that

$$\mathbb{Q}\left[\mathrm{Poly}_d\right] = \mathbb{Q}\left[a_0, a_1, \dots, a_d, a_d^{-1}\right] .$$

Also, the space $\mathrm{Aff} = \{\alpha z + \beta : \alpha \neq 0\}$ of all affine transformations is an algebraic group over $\mathbb{Q}$ under composition. Moreover, Aff acts on Poly_d by conjugation, via $\phi \centerdot f = \phi \circ f \circ \phi^{-1}$. We shall prove the existence of some geometric quotient $\mathcal{P}_d$ of Poly_d by Aff, which is necessarily unique up to isomorphism.

Remark 11. As the algebraic group Aff is not reductive, one cannot directly apply the geometric invariant theory developed in [MFK94] to prove that there exists a geometric quotient $\mathcal{P}_d$ of Poly_d by Aff.

Now, consider the space

$$\mathrm{Poly}_d^{\mathrm{mc}} = \left\{z^d + b_{d-2} z^{d-2} + \dots + b_1 z + b_0\right\}$$

of all monic centered polynomial maps of degree d. Note that $\mathrm{Poly}_d^{\mathrm{mc}}$ is naturally a closed subvariety of Poly_d. Moreover, its stabilizer for the action of Aff on Poly_d by conjugation is the algebraic subgroup $\mu_{d-1} = \left\{\omega : \omega^{d-1} = 1\right\}$ of Aff, under the identification of $\omega \in \mu_{d-1}$ with $\omega z \in \mathrm{Aff}$. Thus, we have an induced action of μ_{d-1} on $\mathrm{Poly}_d^{\mathrm{mc}}$ by conjugation, which is given by

$$\omega \centerdot \left(z^d + \sum_{j=0}^{d-2} b_j z^j\right) = z^d + \sum_{j=0}^{d-2} \omega^{1-j} b_j z^j .$$

Moreover, since μ_{d-1} is finite and $\mathrm{Poly}_d^{\mathrm{mc}}$ is affine, the $\mathbb{Q}$-algebra $\mathbb{Q}\left[\mathrm{Poly}_d^{\mathrm{mc}}\right]^{\mu_{d-1}}$ is finitely generated and the affine variety $\mathcal{P}_d^{\mathrm{mc}}$ over $\mathbb{Q}$ such that

$$\mathbb{Q}\left[\mathcal{P}_d^{\mathrm{mc}}\right] = \mathbb{Q}\left[\mathrm{Poly}_d^{\mathrm{mc}}\right]^{\mu_{d-1}}$$

together with the natural morphism $\pi_d^{\mathrm{mc}} \colon \mathrm{Poly}_d^{\mathrm{mc}} \to \mathcal{P}_d^{\mathrm{mc}}$ is a geometric quotient of $\mathrm{Poly}_d^{\mathrm{mc}}$ by μ_{d-1}, according to Lemma 9.

Finally, we shall apply Lemma 10 to prove the existence of a geometric quotient $\mathcal{P}_d \cong \mathcal{P}_d^{\mathrm{mc}}$ of Poly_d by Aff. To do so, let us first prove the statement below, which implies the well-known fact that each polynomial of degree d over an algebraically closed field of characteristic 0 is conjugate to a monic centered polynomial. Here, we denote by $\theta \colon \mathrm{Aff} \times \mathrm{Poly}_d^{\mathrm{mc}} \to \mathrm{Poly}_d$ the morphism induced by the action of Aff on Poly_d by conjugation.

Claim 12. The morphism $\theta \colon \mathrm{Aff} \times \mathrm{Poly}_d^{\mathrm{mc}} \to \mathrm{Poly}_d$ is finite, flat and surjective.

Proof. For simplicity, write

$$R = \mathbb{Q}\left[\mathrm{Poly}_d\right] = \mathbb{Q}\left[a_0, a_1, \dots, a_d, a_d^{-1}\right]$$

and

$$S = \mathbb{Q}\left[\mathrm{Aff} \times \mathrm{Poly}_d^{\mathrm{mc}}\right] = \mathbb{Q}\left[\alpha, \alpha^{-1}, \beta, b_0, \dots, b_{d-2}\right] .$$

Also denote by K an algebraic closure of the field of fractions $\mathbb{Q}\left(\mathrm{Poly}_d\right)$ of R. For each $\xi \in K$ such that $\xi^{d-1} = a_d$, we have

$$\left(\xi z + \frac{a_{d-1}\xi}{d \cdot a_d}\right) \centerdot \sum_{j=0}^{d} a_j z^j = z^d + \sum_{j=0}^{d-2} B_j^{(\xi)} z^j \in \mathrm{Poly}_d^{\mathrm{mc}}\left(R_\xi\right) ,$$

with

$$R_\xi = \mathbb{Q}\left[a_0, \dots, a_{d-1}, \xi, \xi^{-1}\right] \quad \text{and} \quad B_0^{(\xi)}, \dots, B_{d-2}^{(\xi)} \in R_\xi ,$$

and hence

$$\left(\frac{1}{\xi}z - \frac{a_{d-1}}{d \cdot a_d}\right) \cdot \left(z^d + \sum_{j=0}^{d-2} B_j^{(\xi)} z^j\right) = \sum_{j=0}^{d} a_j z^j \, .$$

In addition, for each $j \in \{0, \dots, d-2\}$ and all $\xi_1, \xi_2 \in K$ such that $\xi_1^{d-1} = a_d$ and $\xi_2^{d-1} = a_d$, we have $H_{\xi_1,\xi_2}\left(B_j^{(\xi_1)}\right) = B_j^{(\xi_2)}$, where $H_{\xi_1,\xi_2} \colon R_{\xi_1} \to R_{\xi_2}$ is the unique $\mathbb{Q}$-algebra homomorphism such that $H_{\xi_1,\xi_2}(a_k) = a_k$ for all $k \in \{0, \dots, d-1\}$ and $H_{\xi_1,\xi_2}(\xi_1) = \xi_2$. Therefore, for each $j \in \{0, \dots, d-2\}$, the polynomial

$$Q_j(T) = \prod_{\xi \in K,\, \xi^{d-1} = a_d} \left(T - B_j^{(\xi)}\right)$$

lies in $R[T]$. For $j \in \{0, \dots, d-2\}$, define $P_j = \theta^*(Q_j) \in S[T]$, where $\theta^* \colon R \to S$ denotes the $\mathbb{Q}$-algebra homomorphism induced by $\theta \colon \operatorname{Aff} \times \operatorname{Poly}_d^{\mathrm{mc}} \to \operatorname{Poly}_d$. Now, consider the closed subvariety Z of $\operatorname{Aff} \times \operatorname{Poly}_d^{\mathrm{mc}}$ given by

$$Z = \left\{\alpha^{d-1} = \theta^*\left(\frac{1}{a_d}\right)\right\} \cap \left\{\beta = \theta^*\left(\frac{-a_{d-1}}{d \cdot a_d}\right)\right\} \cap \bigcap_{j=0}^{d-2} \{P_j(b_j) = 0\} \, .$$

Then the morphism $\theta \colon Z \to \operatorname{Poly}_d$ is finite and surjective. In particular, we have

$$\dim(Z) \geq \dim\left(\operatorname{Poly}_d\right) = d + 1 = \dim\left(\operatorname{Aff} \times \operatorname{Poly}_d^{\mathrm{mc}}\right) \, ,$$

and hence $Z = \operatorname{Aff} \times \operatorname{Poly}_d^{\mathrm{mc}}$ since $\operatorname{Aff} \times \operatorname{Poly}_d^{\mathrm{mc}}$ is irreducible. Thus, the morphism $\theta \colon \operatorname{Aff} \times \operatorname{Poly}_d^{\mathrm{mc}} \to \operatorname{Poly}_d$ is both finite and surjective. Finally, as $\operatorname{Aff} \times \operatorname{Poly}_d^{\mathrm{mc}}$ and Poly_d are both smooth over $\mathbb{Q}$, the morphism θ is also flat by the miracle flatness theorem (see [Mat86, Theorem 23.1]). Thus, the claim is proved. □

By Lemma 10 and Claim 12, the closed immersion $\imath \colon \operatorname{Poly}_d^{\mathrm{mc}} \hookrightarrow \operatorname{Poly}_d$ induces an isomorphism $\imath^* \colon \mathbb{Q}\left[\operatorname{Poly}_d\right]^{\operatorname{Aff}} \to \mathbb{Q}\left[\operatorname{Poly}_d^{\mathrm{mc}}\right]^{\mu_{d-1}}$ of $\mathbb{Q}$-algebras, and thus we have the commutative diagram below.

$$\begin{array}{ccc}
\mathbb{Q}\left[\operatorname{Poly}_d\right]^{\operatorname{Aff}} & \xrightarrow[\sim]{\imath^*} & \mathbb{Q}\left[\operatorname{Poly}_d^{\mathrm{mc}}\right]^{\mu_{d-1}} \\
\big\downarrow & & \big\downarrow \\
\mathbb{Q}\left[\operatorname{Poly}_d\right] & \xrightarrow[\imath^*]{} & \mathbb{Q}\left[\operatorname{Poly}_d^{\mathrm{mc}}\right]
\end{array}$$

Moreover, the affine variety $\mathcal{P}_d$ over $\mathbb{Q}$ given by $\mathbb{Q}[\mathcal{P}_d] = \mathbb{Q}\left[\operatorname{Poly}_d\right]^{\operatorname{Aff}}$ together with the morphism $\pi_d \colon \operatorname{Poly}_d \to \mathcal{P}_d$ induced by the inclusion $\mathbb{Q}\left[\operatorname{Poly}_d\right]^{\operatorname{Aff}} \subseteq \mathbb{Q}\left[\operatorname{Poly}_d\right]$ is a geometric quotient of Poly_d by Aff. This variety $\mathcal{P}_d$ has dimension $d-1$ and is called the *moduli space of polynomial maps* of degree d. We have the commutative diagram below.

$$\begin{array}{ccc}
\operatorname{Poly}_d^{\mathrm{mc}} & \xhookrightarrow{\imath} & \operatorname{Poly}_d \\
{\scriptstyle \pi_d^{\mathrm{mc}}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_d} \\
\mathcal{P}_d^{\mathrm{mc}} & \xrightarrow[\sim]{} & \mathcal{P}_d
\end{array}$$

For $f \in \mathrm{Poly}_d(R)$, with R a commutative $\mathbb{Q}$-algebra, we write $[f] = \pi_d(f) \in \mathcal{P}_d(R)$.

Remark 13. One can explicitly describe the inverse of the $\mathbb{Q}$-algebra isomorphism $\imath^*\colon \mathbb{Q}\left[\mathrm{Poly}_d\right]^{\mathrm{Aff}} \to \mathbb{Q}\left[\mathrm{Poly}_d^{\mathrm{mc}}\right]^{\mu_{d-1}}$. Suppose that $\psi \in \mathbb{Q}\left[\mathrm{Poly}_d^{\mathrm{mc}}\right]^{\mu_{d-1}}$. Write

$$\mathbb{Q}\left[\mathrm{Poly}_d\right] = \mathbb{Q}\left[a_0, a_1, \dots, a_d, a_d^{-1}\right] \quad \text{and} \quad \mathbb{Q}\left[\mathrm{Poly}_d^{\mathrm{mc}}\right] = \mathbb{Q}\left[b_0, \dots, b_{d-2}\right] ,$$

denote by $\overline{\mathbb{Q}\left(\mathrm{Poly}_d\right)}$ the algebraic closure of the field of fractions of $\mathbb{Q}\left[\mathrm{Poly}_d\right]$ and choose any $\xi \in \overline{\mathbb{Q}\left(\mathrm{Poly}_d\right)}$ such that $\xi^{d-1} = a_d$. Now, define

$$\varphi = \psi\left(B_0^{(\xi)}, \dots, B_{d-2}^{(\xi)}\right) ,$$

with $B_0^{(\xi)}, \dots, B_{d-2}^{(\xi)} \in \overline{\mathbb{Q}\left(\mathrm{Poly}_d\right)}$ as in the proof of Claim 12. Then $\varphi \in \mathbb{Q}\left[\mathrm{Poly}_d\right]$. Moreover, φ is the unique element of $\mathbb{Q}\left[\mathrm{Poly}_d\right]^{\mathrm{Aff}}$ such that $\imath^*(\varphi) = \psi$.

Remark 14. For every field K of characteristic 0, the base change Aff_K of Aff to K also acts on the base change $\left(\mathrm{Poly}_d\right)_K$ of Poly_d to K by conjugation, and the base change $\left(\mathcal{P}_d\right)_K$ of $\mathcal{P}_d$ to K is a geometric quotient of $\left(\mathrm{Poly}_d\right)_K$ by Aff_K.

Finally, we shall briefly describe the complex analytic structure of $\mathcal{P}_d(\mathbb{C})$. The set $\mathcal{P}_d(\mathbb{C})$ of complex polynomial maps of degree d modulo conjugation by complex affine transformations is naturally a complex analytic space of dimension $d-1$. In fact, since $\mathrm{Poly}_d(\mathbb{C}) \cong \mathbb{C}^d \times \mathbb{C}^*$ is a complex manifold and the action of $\mathrm{Aff}(\mathbb{C})$ on $\mathrm{Poly}_d(\mathbb{C})$ by conjugation is proper, faithful and its stabilizers are all finite, $\mathcal{P}_d(\mathbb{C})$ is a complex orbifold. Moreover, $\mathcal{P}_d^{\mathrm{mc}}(\mathbb{C})$ is also a complex orbifold, and we have a natural biholomorphism $\mathcal{P}_d(\mathbb{C}) \cong \mathcal{P}_d^{\mathrm{mc}}(\mathbb{C})$. The complex topology of $\mathcal{P}_d(\mathbb{C})$ is the quotient topology: any subset U of $\mathcal{P}_d(\mathbb{C})$ is open if and only if $\pi_d^{-1}(U)$ is an open subset of $\mathrm{Poly}_d(\mathbb{C})$. We refer to [Car22] for further information about orbifolds.

We say that a sequence $(f_n)_{n\geq 0}$ of elements of $\mathrm{Poly}_d(\mathbb{C})$ *degenerates* in $\mathcal{P}_d(\mathbb{C})$ if, for every compact subset K of $\mathcal{P}_d(\mathbb{C})$, we have $[f_n] \in \mathcal{P}_d(\mathbb{C}) \setminus K$ for all sufficiently large n. Thus, any sequence $(f_n)_{n\geq 0}$ of elements of $\mathrm{Poly}_d(\mathbb{C})$ degenerates in $\mathcal{P}_d(\mathbb{C})$ if and only if there does not exist any sequence $(\phi_n)_{n\geq 0}$ of elements of $\mathrm{Aff}(\mathbb{C})$ such that $(\phi_n \centerdot f_n)_{n\geq 0}$ has a convergent subsequence in $\mathrm{Poly}_d(\mathbb{C})$. We can also express degeneration in $\mathcal{P}_d(\mathbb{C})$ in terms of maximal escape rates. Explicitly, any sequence $(f_n)_{n\geq 0}$ of elements of $\mathrm{Poly}_d(\mathbb{C})$ degenerates in the moduli space $\mathcal{P}_d(\mathbb{C})$ if and only if $\lim\limits_{n\to+\infty} M_{f_n} = +\infty$ (see [BH88, Proposition 3.6]).

2.2. **Multiplier spectrum morphisms.** Now, let us give a precise definition of the morphisms $\mathrm{Mult}_d^{(P)}$, with $P \geq 1$. To do this, we shall first recall the notions of dynatomic and multiplier polynomials associated with a polynomial map. We refer the reader to [MP94] and [VH92] for further details.

Suppose that R is any $\mathbb{Q}$-algebra that is an integral domain and $f \in \mathrm{Poly}_d(R)$. Then there is a unique sequence $\left(\Phi_f^{(p)}\right)_{p\geq 1}$ of elements of $R[z]$ such that, for each $p \geq 1$, we have

$$f^{\circ p}(z) - z = \prod_{k \mid p} \Phi_f^{(k)}(z) \, .$$

For $p \geq 1$, the polynomial $\Phi_f^{(p)} \in R[z]$ is called the pth *dynatomic polynomial* of f. For every $p \geq 1$, we have $\deg\left(\Phi_f^{(p)}\right) = \nu_d^{(p)}$, where

$$\nu_d^{(p)} = \sum_{k \mid p} \mu\left(\frac{p}{k}\right) d^k$$

and $\mu\colon \mathbb{Z}_{\geq 1} \to \{-1, 0, 1\}$ denotes the Möbius function.

The result below gives the relation between the periodic points of a polynomial map and its dynatomic polynomials.

Proposition 15 ([MS95, Proposition 3.2])**.** *Assume that R is any $\mathbb{Q}$-algebra that is an integral domain, $f \in \mathrm{Poly}_d(R)$ and $p \geq 1$. Then $z_0 \in R$ is a root of $\Phi_f^{(p)}$ if and only if either z_0 is a periodic point for f with period p or z_0 is a periodic point for f with period a proper divisor k of p and multiplier a primitive $\frac{p}{k}$th root of unity.*

Suppose that R is any $\mathbb{Q}$-algebra that is an integral domain and $f \in \mathrm{Poly}_d(R)$. For every $p \geq 1$, there exists a unique monic polynomial $\chi_f^{(p)} \in R[\lambda]$ such that

$$\chi_f^{(p)}(\lambda)^p = a_d^{-m_d^{(p)}} \operatorname{res}_z\left(\Phi_f^{(p)}(z), \lambda - \left(f^{\circ p}\right)'(z)\right),$$

where $a_d \in R^*$ denotes the leading coefficient of f, res_z denotes the resultant with respect to z and

$$m_d^{(p)} = \begin{cases} d-1 & \text{if } p = 1 \\ \frac{\nu_d^{(p)}(d^p-1)}{d-1} & \text{if } p \geq 2 \end{cases}.$$

For $p \geq 1$, the polynomial $\chi_f^{(p)} \in R[\lambda]$ is called the pth *multiplier polynomial* of f. For every $p \geq 1$, we have $\deg\left(\chi_f^{(p)}\right) = N_d^{(p)}$, where $N_d^{(p)} = \frac{\nu_d^{(p)}}{p}$.

For an algebraically closed field K of characteristic 0, $f \in \mathrm{Poly}_d(K)$ and $p \geq 1$, we denote by $\Lambda_f^{(p)} \in K^{N_d^{(p)}}/\mathfrak{S}_{N_d^{(p)}}$ the multiset of roots of $\chi_f^{(p)}$.

Using Proposition 15, we immediately obtain the result below, which relates the multiplier polynomials of a polynomial map to its multipliers.

Proposition 16. *Assume that K is an algebraically closed field of characteristic 0, $f \in \mathrm{Poly}_d(K)$ and $p \geq 1$. Then $\lambda \in K$ lies in $\Lambda_f^{(p)}$ if and only if at least one of the following two conditions is satisfied:*

- *λ is a multiplier of f at a cycle with period p;*
- *$\lambda = 1$ and f has a cycle with period a proper divisor k of p and multiplier a primitive $\frac{p}{k}$th root of unity.*

In particular, if f has no parabolic cycle with period dividing p, then $\Lambda_f^{(p)}$ consists precisely of the multipliers of f at its cycles with period p.

Now, consider the generic polynomial

$$\boldsymbol{f}(z) = \sum_{j=0}^{d} a_j z^j \in \mathrm{Poly}_d\left(\mathbb{Q}\left[\mathrm{Poly}_d\right]\right).$$

For $p \geq 1$, write

$$\chi_{\boldsymbol{f}}^{(p)}(\lambda) = \lambda^{N_d^{(p)}} + \sum_{j=1}^{N_d^{(p)}} (-1)^j \boldsymbol{\sigma}_{d,j}^{(p)} \lambda^{N_d^{(p)}-j} \in \mathbb{Q}\left[\mathrm{Poly}_d\right][\lambda]\,.$$

Specializing, for every $\mathbb{Q}$-algebra R that is an integral domain, every $f \in \mathrm{Poly}_d(R)$ and every $p \geq 1$, we have

$$\chi_f^{(p)}(\lambda) = \lambda^{N_d^{(p)}} + \sum_{j=1}^{N_d^{(p)}} (-1)^j \boldsymbol{\sigma}_{d,j}^{(p)}(f) \lambda^{N_d^{(p)}-j}\,.$$

Thus, for every algebraically closed field K of characteristic 0, every $f \in \mathrm{Poly}_d(K)$ and every $p \geq 1$, the $\sigma_{d,j}^{(p)}(f)$, with $j \in \left\{1, \dots, N_d^{(p)}\right\}$, are the elementary symmetric functions of the elements of $\Lambda_f^{(p)}$. As the multiplier is invariant under conjugation, it follows that the regular function $\boldsymbol{\sigma}_{d,j}^{(p)} \in \mathbb{Q}\left[\mathrm{Poly}_d\right]$ is invariant under the action of Aff on Poly_d by conjugation for each $p \geq 1$ and each $j \in \left\{1, \dots, N_d^{(p)}\right\}$. Therefore, for every $p \geq 1$ and every $j \in \left\{1, \dots, N_d^{(p)}\right\}$, there exists a unique regular function $\sigma_{d,j}^{(p)} \in \mathbb{Q}\left[\mathcal{P}_d\right]$ such that $\sigma_{d,j}^{(p)}\left([f]\right) = \boldsymbol{\sigma}_{d,j}^{(p)}(f)$ for each commutative $\mathbb{Q}$-algebra R and each $f \in \mathrm{Poly}_d(R)$. For $P \geq 1$, we define the *multiplier spectrum morphism*

$$\mathrm{Mult}_d^{(P)} = \left(\left(\sigma_{d,j}^{(1)}\right)_{1 \leq j \leq N_d^{(1)}}, \dots, \left(\sigma_{d,j}^{(P)}\right)_{1 \leq j \leq N_d^{(P)}}\right) \colon \mathcal{P}_d \to \prod_{p=1}^{P} \mathbb{A}^{N_d^{(p)}}\,.$$

For $P \geq 1$, we denote by $\Sigma_d^{(P)}$ the scheme-theoretic image of $\mathrm{Mult}_d^{(P)}$, which equals the Zariski-closure of $\mathrm{Mult}_d^{(P)}\left(\mathcal{P}_d(\mathbb{Q})\right)$ in $\prod_{p=1}^{P} \mathbb{A}^{N_d^{(p)}}$.

Finally, we shall recall a few facts about the multipliers at the fixed points. For every algebraically closed field K of characteristic 0 and every $f \in \mathrm{Poly}_d(K)$ such that $\lambda \neq 1$ for all $\lambda \in \Lambda_f^{(1)}$, we have

$$\sum_{\lambda \in \Lambda_f^{(1)}} \frac{1}{1-\lambda} = 0\,.$$

The relation above is known as the holomorphic fixed-point formula. Therefore, in $\mathbb{Q}\left[\mathcal{P}_d\right]$, we have

$$d + \sum_{j=1}^{d} (-1)^j (d-j) \sigma_{d,j}^{(1)} = 0\,.$$

In fact, denoting by $s_1, \dots, s_d$ the standard coordinates on $\mathbb{A}^d$, we have

$$\Sigma_d^{(1)} = \left\{ d + \sum_{j=1}^{d} (-1)^j (d-j) s_j = 0 \right\} \subseteq \mathbb{A}^d\,,$$

and Fujimura showed in [Fuj07] that the morphism $\mathrm{Mult}_d^{(1)} \colon \mathcal{P}_d \to \Sigma_d^{(1)}$ has degree $(d-2)!$. In addition, Fujimura also proved that the morphism $\mathrm{Mult}_d^{(1)} \colon \mathcal{P}_d \to \Sigma_d^{(1)}$ is neither surjective nor quasifinite when $d \geq 4$. We refer the reader to Sugiyama's articles [Sug17] and [Sug23] for the exact number of conjugacy classes $[f] \in \mathcal{P}_d(\mathbb{C})$ that satisfy $\Lambda_f^{(1)} = \Lambda$, for each $\Lambda \in \mathbb{C}^d/\mathfrak{S}_d$ such that $\lambda \neq 1$ for all $\lambda \in \Lambda$.

2.3. **The cases of quadratic and cubic polynomial maps.** To conclude this section, let us briefly describe the moduli spaces $\mathcal{P}_2$ and $\mathcal{P}_3$ of quadratic and cubic polynomial maps, respectively, and show that the morphisms $\mathrm{Mult}_2^{(1)}$ and $\mathrm{Mult}_3^{(1)}$ induced by the multipliers at the fixed points are isomorphisms onto their images. The results presented here are well known.

Example 17. We first study the case of quadratic polynomial maps. Since we have a natural isomorphism $\mathcal{P}_2 \cong \mathcal{P}_2^{\mathrm{mc}}$, we may restrict our attention to monic centered quadratic polynomials. As $\mu_1 = \{1\}$ is the trivial algebraic group, we have

$$\mathcal{P}_2^{\mathrm{mc}} \cong \mathrm{Poly}_2^{\mathrm{mc}} = \left\{z^2 + a_0\right\} \quad \text{and} \quad \mathbb{Q}\left[\mathcal{P}_2^{\mathrm{mc}}\right] = \mathbb{Q}\left[\mathrm{Poly}_2^{\mathrm{mc}}\right] = \mathbb{Q}\left[a_0\right] .$$

Now, computing the polynomial $\chi_f^{(1)} \in \mathbb{Q}\left[\mathcal{P}_2^{\mathrm{mc}}\right][\lambda]$ for $f(z) = z^2 + a_0 \in \mathrm{Poly}_2^{\mathrm{mc}}$, we obtain

$$\sigma_{2,1}^{(1)} = 2 \in \mathbb{Q}\left[\mathcal{P}_2^{\mathrm{mc}}\right] \quad \text{and} \quad \sigma_{2,2}^{(1)} = 4a_0 \in \mathbb{Q}\left[\mathcal{P}_2^{\mathrm{mc}}\right]$$

via the natural isomorphism $\mathcal{P}_2 \cong \mathcal{P}_2^{\mathrm{mc}}$. Therefore, we have

$$\mathbb{Q}\left[\Sigma_2^{(1)}\right] = \mathbb{Q}\left[\sigma_{2,1}^{(1)}, \sigma_{2,2}^{(1)}\right] = \mathbb{Q}\left[a_0\right] = \mathbb{Q}\left[\mathcal{P}_2^{\mathrm{mc}}\right] ,$$

where $\Sigma_2^{(1)}$ denotes the image of the morphism $\mathrm{Mult}_2^{(1)} = \left(\sigma_{2,1}^{(1)}, \sigma_{2,2}^{(1)}\right) : \mathcal{P}_2^{\mathrm{mc}} \to \mathbb{A}^2$. Thus, $\mathrm{Mult}_2^{(1)}$ induces an isomorphism from $\mathcal{P}_2$ onto its image $\Sigma_2^{(1)}$.

Example 18. We now turn to the case of cubic polynomial maps. A similar discussion can be found in [Mil92, Appendix A]. As $\mathcal{P}_3 \cong \mathcal{P}_3^{\mathrm{mc}}$, we restrict our attention to monic centered cubic polynomials. Recall that

$$\mathrm{Poly}_3^{\mathrm{mc}} = \left\{z^3 + a_1 z + a_0\right\}$$

and that the algebraic group $\mu_2 = \{\pm 1\}$ acts on $\mathrm{Poly}_3^{\mathrm{mc}}$ by

$$\omega \centerdot \left(z^3 + a_1 z + a_0\right) = z^3 + a_1 z + \omega a_0 .$$

Therefore, we have

$$\mathbb{Q}\left[\mathcal{P}_3^{\mathrm{mc}}\right] = \mathbb{Q}\left[\mathrm{Poly}_3^{\mathrm{mc}}\right]^{\mu_2} = \mathbb{Q}[\alpha, \beta] , \quad \text{with} \quad \alpha = a_1 \quad \text{and} \quad \beta = a_0^2 .$$

Now, for simplicity, write $s_j = \sigma_{3,j}^{(1)}$ for $j \in \{1, 2, 3\}$. Computing $\chi_f^{(1)} \in \mathbb{Q}\left[\mathcal{P}_3^{\mathrm{mc}}\right][\lambda]$ for $f(z) = z^3 + a_1 z + a_0 \in \mathrm{Poly}_3^{\mathrm{mc}}$, we obtain

$$s_1 = -3\alpha + 6 , \quad s_2 = -6\alpha + 9 \quad \text{and} \quad s_3 = 4\alpha^3 - 12\alpha^2 + 9\alpha + 27\beta$$

via the natural isomorphism $\mathcal{P}_3 \cong \mathcal{P}_3^{\mathrm{mc}}$, which yields

$$\alpha = \frac{-1}{3} s_1 + 2 \quad \text{and} \quad \beta = \frac{4}{729} s_1^3 - \frac{4}{81} s_1^2 + \frac{1}{9} s_1 + \frac{1}{27} s_3 - \frac{2}{27} .$$

Therefore, we have

$$\mathbb{Q}\left[\Sigma_3^{(1)}\right] = \mathbb{Q}\left[s_1, s_2, s_3\right] = \mathbb{Q}[\alpha, \beta] = \mathbb{Q}\left[\mathcal{P}_3^{\mathrm{mc}}\right] ,$$

where $\Sigma_3^{(1)}$ denotes the image of the morphism $\mathrm{Mult}_3^{(1)} = (s_1, s_2, s_3) : \mathcal{P}_3^{\mathrm{mc}} \to \mathbb{A}^3$. Thus, $\mathrm{Mult}_3^{(1)}$ induces an isomorphism from $\mathcal{P}_3$ onto its image $\Sigma_3^{(1)}$.

3. Multipliers at small cycles and maximal escape rates for complex polynomial maps

In this section, we shall first prove Theorem B in the complex case. Our proof is inspired by the article [DM08] by DeMarco and McMullen. It relies on a combinatorial argument and on an inequality relating the modulus of the multiplier at a repelling periodic point and the modulus of some annulus. As the latter is already known, the main novelty here is a combinatorial result concerning sublevel sets of the Green function of a polynomial map with disconnected Julia set. Nonetheless, we provide a detailed proof of Theorem B in the complex setting for completeness and to exhibit the similarity with our proof in the non-Archimedean case. Finally, we shall close this section by deriving Corollaries A.1 and A.2 from Theorem A.

We fix here an integer $d \geq 2$. In this section and in the next one, we sometimes use the letter e to denote the degree of certain maps, but we never write e for the exponential function, which we denote by exp.

3.1. **The Green function of a complex polynomial map.** First, let us recall some well-known facts regarding the Green function of a complex polynomial map. We refer to [CG93, Chapter III, Section 4] and [DH84, Exposé VIII, Section I] for further information.

Suppose that $f \in \mathrm{Poly}_d(\mathbb{C})$. Recall that the *filled Julia set* $\mathcal{K}_f$ of f is given by

$$\mathcal{K}_f = \left\{ z \in \mathbb{C} : \sup_{n \geq 0} \left| f^{\circ n}(z) \right| < +\infty \right\} .$$

Also recall that the *Green function* $g_f \colon \mathbb{C} \to \mathbb{R}_{\geq 0}$ of f is given by

$$g_f(z) = \lim_{n \to +\infty} \frac{1}{d^n} \log^+ \left| f^{\circ n}(z) \right| .$$

This map g_f is well defined, continuous and subharmonic on $\mathbb{C}$ and it is harmonic on $\mathbb{C} \setminus \mathcal{K}_f$. Moreover, we have $g_f \circ f = d \cdot g_f$ and $g_f(z) = \log|z| + O(1)$ as $z \to \infty$, and in particular $\{g_f = 0\} = \mathcal{K}_f$. Define the *maximal escape rate* M_f of f by

$$M_f = \max \left\{ g_f(c) : c \in \mathbb{C},\ f'(c) = 0 \right\} .$$

It follows from the Riemann–Hurwitz formula that the set $\mathcal{K}_f$ is connected if and only if $M_f = 0$ or, equivalently, if and only if the critical points for f all lie in $\mathcal{K}_f$ (see [Bea91, Theorem 9.5.1]).

Note that f is conjugate to $z \mapsto z^d$ near infinity by Böttcher's theorem since ∞ is a fixed point for f with local degree d, viewing f as a rational map $f \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. In fact, there exists a biholomorphism

$$\phi_f \colon \{g_f > M_f\} \to \mathbb{C} \setminus \overline{D\left(0, \exp\left(M_f\right)\right)}$$

such that $\lim_{z \to \infty} \phi_f(z) = \infty$ and $\phi_f \circ f = \phi_f^d$. This biholomorphism ϕ_f is unique up to multiplication by a $(d-1)$th root of unity and is called a *Böttcher coordinate* of f at infinity. Moreover, we have $g_f = \log|\phi_f|$.

Now, we say that a compact subset K of $\mathbb{C}$ is *full* if the set $\mathbb{C} \setminus K$ is connected. It follows from the maximum principle that $\{g_f \leq \eta\}$ is a full compact subset of $\mathbb{C}$ for all $\eta \in \mathbb{R}_{\geq 0}$. In addition, $\{g_f < \eta\}$ is the interior of $\{g_f \leq \eta\}$ for all $\eta \in \mathbb{R}_{>0}$ as g_f has no local maximum on $\mathbb{C} \setminus \mathcal{K}_f$. Therefore, for every $\eta \in \mathbb{R}_{>0}$, the connected components of $\{g_f < \eta\}$ are all bounded simply connected subsets of $\mathbb{C}$. Moreover, for every $\eta \in \mathbb{R}_{>0}$, the connected components of $\{g_f < \eta\}$ all intersect $\mathcal{K}_f$ since g_f has no local minimum on $\mathbb{C} \setminus \mathcal{K}_f$.

For each $\eta \in [M_f, +\infty)$, the set $\{g_f \leq \eta\}$ is connected as $\widehat{\mathbb{C}} \setminus \{g_f \leq \eta\}$ is biholomorphic to $\widehat{\mathbb{C}} \setminus \overline{D(0, \exp(\eta))}$ under any Böttcher coordinate ϕ_f of f at infinity. As a result, $\{g_f < \eta\}$ is also connected for all $\eta \in (M_f, +\infty)$. In contrast, we have the following:

Lemma 19. *Suppose that $f \in \mathrm{Poly}_d(\mathbb{C})$ has a disconnected filled Julia set $\mathcal{K}_f$, and define $C \geq 1$ to be the number of critical points $c \in \mathbb{C}$ for f such that $g_f(c) = M_f$, counting multiplicities. Then $\{g_f < M_f\}$ has exactly $C + 1$ connected components.*

Proof. Denote here by $U_1, \ldots, U_N$ the connected components of $\{g_f < M_f\}$, with $N \geq 1$. For $j \in \{1, \ldots, N\}$, denote by $C_j \geq 0$ the number of critical points for f in U_j, counting multiplicities. Note that $\sum_{j=1}^{N} C_j = d - 1 - C$. Now, as

$$\{g_f < M_f\} = f^{-1}\left(\{g_f < d \cdot M_f\}\right) ,$$

the map $f \colon U_j \to \{g_f < d \cdot M_f\}$ is proper of degree $d_j \geq 1$ for each $j \in \{1, \ldots, N\}$, and we have $d = \sum_{j=1}^{N} d_j$. In addition, for each $j \in \{1, \ldots, N\}$, we have $d_j = C_j + 1$ by the Riemann–Hurwitz formula because U_j is simply connected by the previous discussion. Therefore, we have

$$d = \sum_{j=1}^{N} (C_j + 1) = d - 1 - C + N ,$$

and hence $N = C + 1$. Thus, the lemma is proved. □

Remark 20. In fact, we shall only use the well-known fact that, if $f \in \mathrm{Poly}_d(\mathbb{C})$ has a disconnected filled Julia set $\mathcal{K}_f$, then $\{g_f < M_f\}$ is disconnected.

3.2. **A combinatorial argument.** Now, let us count the critical points in certain sublevel sets of the Green function to obtain a result implying that the Julia set of any polynomial map either is connected or has a connected component consisting only of a periodic point with period 1 or 2. We shall present this as a consequence of a general two-islands lemma.

To obtain our two-islands lemma, we shall prove a result regarding preimages of simply connected domains under holomorphic maps. To do so, we shall first prove the general fact below.

Lemma 21. *Suppose that X is a topological space that is both connected and locally connected, A is a connected subset of X and B is a clopen subset of $X \setminus A$. Then $A \cup B$ is connected.*

Proof. Note that the desired result is immediate if $A = \varnothing$. From now on, suppose that $A \neq \varnothing$. Then it suffices to prove that $A \cup C$ is connected for each connected component C of B. Thus, assume that C is a connected component of B. Denote by D the connected component of $X \setminus A$ containing C. Then $B \cap D = D$ because $B \cap D$ is a nonempty clopen subset of D and D is connected, which yields $D \subseteq B$, and hence $D = C$. Thus, C is a connected component of $X \setminus A$, and in particular

$$\partial C \subseteq \partial(X \setminus A) = \partial A$$

since X is locally connected. Furthermore, $\partial C \neq \varnothing$ since, otherwise, C would be a nonempty clopen subset of X contained in $X \setminus A$. Choose $x \in \partial C$. If $x \in A$, then $x \in A \cap \overline{C}$. Now, suppose that $x \in X \setminus A$. Denote by C' the connected component of $X \setminus A$ containing x. Then $C \cup C'$ is connected because C and C' are connected and $x \in \overline{C} \cap C'$, and hence $C = C'$. As a result, $x \in \overline{A} \cap C$. Thus, we have proved that $A \cap \overline{C} \neq \varnothing$ or $\overline{A} \cap C \neq \varnothing$. As A and C are connected, it follows that $A \cup C$ is connected. This completes the proof of the lemma. □

Using the previous lemma, we obtain the general result below, which the author was unable to find in the literature.

Lemma 22. *Suppose that U, V are nonempty simply connected open subsets of $\mathbb{C}$ and $f \colon U \to \mathbb{C}$ is a holomorphic map. Then every connected component of $f^{-1}(V)$ is simply connected.*

Proof. Note that the desired result is immediate if f is constant. Assume now that f is not constant. Recall that a connected open subset D of $\mathbb{C}$ is simply connected if and only if $\widehat{\mathbb{C}} \setminus D$ is connected, where $\widehat{\mathbb{C}}$ is the Riemann sphere. Suppose that U_0 is a connected component of $f^{-1}(V)$, and let us show that U_0 is simply connected. Thus, assume that A is a clopen subset of $\widehat{\mathbb{C}} \setminus U_0$ that does not contain ∞, and let us prove that $A = \varnothing$. Note that $A \setminus U$ is a clopen subset of $\widehat{\mathbb{C}} \setminus U$, which does not contain ∞, and hence $A \subseteq U$ since $U \subseteq \mathbb{C}$ is simply connected. Now, define

$$B = f\left(A \cup U_0\right) \setminus V .$$

Then B is an open subset of $\widehat{\mathbb{C}} \setminus V$ since $A \cup U_0 = \widehat{\mathbb{C}} \setminus \left(\left(\widehat{\mathbb{C}} \setminus U_0\right) \setminus A\right)$ is an open subset of U and the map f is open. Moreover, we have

$$B = f\left(A \setminus f^{-1}(V)\right)$$

and $A \setminus f^{-1}(V)$ is a compact subset of U since it is closed in $\widehat{\mathbb{C}}$. It follows that B is compact, and in particular it is also closed in $\widehat{\mathbb{C}} \setminus V$. Therefore, $B = \varnothing$ because $V \subseteq \mathbb{C}$ is simply connected and $B \subseteq \mathbb{C}$, and hence $A \subseteq f^{-1}(V)$. Moreover, $A \cup U_0$ is connected by Lemma 21. Therefore, $A \subseteq U_0$ since U_0 is a connected component of $f^{-1}(V)$, and hence $A = \varnothing$. This completes the proof of the lemma. □

Finally, counting critical points, we deduce the two-islands lemma below. This statement, which greatly simplified the author's exposition, was communicated to him by Buff. For comparison, we refer the reader to [Ber00] for information about the classical Ahlfors five-islands theorem.

Lemma 23. *Suppose that U, V are nonempty simply connected open subsets of $\mathbb{C}$, $f \colon U \to V$ is a proper holomorphic map and V_1, V_2 are disjoint nonempty simply connected open subsets of V. Then there exist an index $j \in \{1, 2\}$ and a connected component U_j of $f^{-1}\left(V_j\right)$ such that f induces a biholomorphism from U_j to V_j.*

Proof. Denote by $e \geq 1$ the degree of $f \colon U \to V$. For $j \in \{1, 2\}$, denote by $C_j \geq 0$ the number of critical points for f in $f^{-1}\left(V_j\right)$, counting multiplicities. Then

$$e = C + 1 \geq C_1 + C_2 + 1 \geq 2 \min\left\{C_1, C_2\right\} + 1$$

by the Riemann–Hurwitz formula, where $C \geq 0$ is the number of critical points for f in U, counting multiplicities. Now, for $j \in \{1, 2\}$, denote by $U_j^{(1)}, \dots, U_j^{(N_j)}$, with $N_j \geq 1$, the connected components of $f^{-1}\left(V_j\right)$. Then, for each $j \in \{1, 2\}$ and each

$\ell \in \{1, \dots, N_j\}$, the map $f \colon U_j^{(\ell)} \to V_j$ is proper of degree $e_j^{(\ell)} \geq 1$. For $j \in \{1, 2\}$, we have $e = \sum\limits_{\ell=1}^{N_j} e_j^{(\ell)}$. Moreover, $U_j^{(\ell)}$ is a simply connected open subset of $\mathbb{C}$ for all $j \in \{1, 2\}$ and all $\ell \in \{1, \dots, N_j\}$ by Lemma 22. By the Riemann–Hurwitz formula, it follows that $e_j^{(\ell)} = C_j^{(\ell)} + 1$, where $C_j^{(\ell)} \geq 0$ is the number of critical points for f in $U_j^{(\ell)}$, counting multiplicities, for all $j \in \{1, 2\}$ and all $\ell \in \{1, \dots, N_j\}$. Thus,

$$\forall j \in \{1, 2\},\, e = \sum_{\ell=1}^{N_j} \left(C_j^{(\ell)} + 1 \right) = C_j + N_j \,.$$

Therefore, there exists $j \in \{1, 2\}$ such that $C_j < N_j$ as, otherwise, we would have $e \leq 2 \min \{C_1, C_2\}$. Then $C_j^{(\ell)} = 0$ for some $\ell \in \{1, \dots, N_j\}$, and we have $e_j^{(\ell)} = 1$. Thus, f induces a biholomorphism from $U_j^{(\ell)}$ to V_j, and the lemma is proved. □

Remark 24. In fact, under the hypotheses of Lemma 23, one of the following two conditions is satisfied:

(1) there exist an index $j \in \{1, 2\}$ and distinct connected components $U_j^{(1)}$ and $U_j^{(2)}$ of $f^{-1}(V_j)$ that are both mapped biholomorphically onto V_j by f;
(2) there exist connected components U_1 and U_2 of $f^{-1}(V_1)$ and $f^{-1}(V_2)$ that are mapped biholomorphically onto V_1 and V_2 by f, respectively.

Returning to our proof of Lemma 23, the condition (1) is satisfied if $C_1 \neq C_2$ and the condition (2) is satisfied if $C_1 = C_2$.

Applying the previous lemma to a dynamical setting, we easily obtain the result below, which is a key ingredient in our proof of Theorem B in the complex case.

Lemma 25. *Suppose that $f \in \mathrm{Poly}_d(\mathbb{C})$ has a disconnected filled Julia set $\mathcal{K}_f$. Then one of the following two conditions is satisfied:*

(1) *there exists a connected component U of $\left\{ g_f < \frac{M_f}{d} \right\}$ such that f induces a biholomorphism from U onto the connected component V of $\{g_f < M_f\}$ containing U;*
(2) *for all distinct connected components V, V' of $\{g_f < M_f\}$, there exists a connected component U of $\left\{ g_f < \frac{M_f}{d} \right\}$ contained in V such that f induces a biholomorphism from U onto V'.*

In addition, if $d \in \{2, 3\}$, then there exists a connected component V of $\{g_f < M_f\}$ such that f induces a biholomorphism from V onto $\{g_f < d \cdot M_f\}$.

Proof. If the condition (1) is satisfied, we are done. Now, suppose that this is not the case, and let us show that the condition (2) is satisfied. Assume that V, V' are two distinct connected components of $\{g_f < M_f\}$. Then $f \colon V \to \{g_f < d \cdot M_f\}$ is a proper map and there is no connected component of $\left\{ g_f < \frac{M_f}{d} \right\}$ contained in V that is mapped biholomorphically onto V by f. As a result, by Lemma 23, f maps a connected component of $\left\{ g_f < \frac{M_f}{d} \right\}$ contained in V biholomorphically onto V'. Thus, the desired result is proved.

Now, assume that $d \in \{2, 3\}$. Denote by $V_1, \dots, V_N$, with $N \geq 2$, the connected components of $\{g_f < M_f\}$. For $j \in \{1, \dots, N\}$, the map $f \colon V_j \to \{g_f < d \cdot M_f\}$ is proper of degree $d_j \geq 1$. We have $d = \sum_{j=1}^{N} d_j$. Therefore, as $N \geq 2$ and $d \leq 3$, there exists $j \in \{1, \dots, N\}$ such that $d_j = 1$. Thus, f induces a biholomorphism from V_j onto $\{g_f < d \cdot M_f\}$, and the lemma is proved. □

Remark 26. As the connected components of $\{g_f < \eta\}$, with $\eta \in \mathbb{R}_{>0}$, are known to be simply connected, our proof of Lemma 25 does not actually require Lemma 22 and one could have simply added as an assumption in the statement of Lemma 23 the fact that the connected components of $f^{-1}(V_j)$, with $j \in \{1, 2\}$, are all simply connected. Nonetheless, Lemmas 22 and 23 are of interest in their own right.

Remark 27. Using Remark 24, one can replace the condition (1) in the statement of Lemma 25 by the following one:

(1') there exist a connected component V of $\{g_f < M_f\}$ and distinct connected components U, U' of $\left\{g_f < \frac{M_f}{d}\right\}$ contained in V such that f maps both U and U' biholomorphically onto V.

As a consequence of Lemma 25, we easily obtain the result below, which is not used in our proof of Theorem B but may be of independent interest.

Proposition 28. *Assume that $f \in \mathrm{Poly}_d(\mathbb{C})$ has a disconnected filled Julia set $\mathcal{K}_f$. Then $\mathcal{K}_f$ has a connected component that consists only of a periodic point for f with period* 1 *or* 2*. Furthermore, if $d \in \{2, 3\}$, then $\mathcal{K}_f$ has a connected component that consists only of a fixed point for f.*

Proof. Assume for a moment that the condition (2) of Lemma 25 holds. Denote by $V_1, \dots, V_N$, with $N \geq 2$, the connected components of $\{g_f < M_f\}$. Then there are connected components U_1, U_2 of $\left\{g_f < \frac{M_f}{d}\right\}$ contained in V_1, V_2, respectively, such that f induces biholomorphisms from U_1 to V_2 and from U_2 to V_1. Define g_1 to be the inverse of $f \colon U_1 \to V_2$. Then $g_1(U_1)$ is a connected component of $\left\{g_f < \frac{M_f}{d^2}\right\}$ contained in U_1 and the map $f^{\circ 2} \colon g_1(U_1) \to V_1$ is a biholomorphism.

Thus, by Lemma 25, there exist $\eta \in \mathbb{R}_{>0}$, $p \in \{1, 2\}$ and a connect component U of $\left\{g_f < \frac{\eta}{d^p}\right\}$ such that $f^{\circ p}$ induces a biholomorphism from U onto the connected component V of $\{g_f < \eta\}$ containing U, and we can take $p = 1$ if $d \in \{2, 3\}$. Now, denote by g the inverse of $f^{\circ p} \colon U \to V$, and define

$$S = \bigcap_{n \geq 0} g^{\circ n}(U) \, .$$

Let us show that S is a connected component of $\mathcal{K}_f$, which consists only of a fixed point for $f^{\circ p}$. Note that $S \subseteq \mathcal{K}_f$ because U is bounded. Moreover, S is nonempty and connected since $S = \bigcap_{n \geq 0} g^{\circ n}\left(\overline{U}\right)$ is the intersection of a decreasing sequence of nonempty, connected and compact subsets of $\mathbb{C}$. Now, denote by C the connected component of $\mathcal{K}_f$ containing S. Then, for every $n \geq 0$, we have

$$f^{\circ pn}(C) \subseteq \mathcal{K}_f \subseteq \left\{g_f < \frac{\eta}{d^p}\right\} \quad \text{and} \quad \varnothing \neq f^{\circ pn}(S) \subseteq f^{\circ pn}(C) \cap U \, ,$$

and hence $f^{\circ pn}(C) \subseteq U$ as $f^{\circ pn}(C)$ is connected and U is a connected component of $\left\{g_f < \frac{\eta}{d^p}\right\}$. It follows by induction that

$$\forall n \geq 0, \, C = g^{\circ n}\left(f^{\circ pn}(C)\right) \subseteq g^{\circ n}(U) \, .$$

Thus, $C = S$. Now, note that g is contracting with respect to the Poincaré metric on V by the Schwarz lemma, since $U \Subset V$. Therefore, S consists of a single point, which is necessarily fixed for $f^{\circ p}$ because S is invariant under $f^{\circ p}$. This completes the proof of the proposition. □

Remark 29. Note that, if $f \in \mathrm{Poly}_d(\mathbb{C})$ has a disconnected filled Julia set $\mathcal{K}_f$, then $\mathcal{K}_f$ has uncountably many connected components. In [QY09], Qiu and Yin proved that, for every $f \in \mathrm{Poly}_d(\mathbb{C})$, all but countably many connected components of $\mathcal{K}_f$ consist of a single point. In contrast, as shown by McMullen in [McM88], there are rational maps $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ whose Julia set $\mathcal{J}_f$ is disconnected but has no connected component that consists of a single point. The existence of connected components of the Julia set consisting of a single point for transcendental maps was studied by Domínguez in [Dom97] and [Dom98].

Remark 30. In fact, using Remark 27, one can show that, for every $f \in \mathrm{Poly}_d(\mathbb{C})$, either $\mathcal{K}_f$ is connected or $\mathcal{K}_f$ has a connected component consisting only of a periodic point for f with period 2.

3.3. **Multipliers and maximal escape rates.** Here, let us obtain lower bounds on moduli on multipliers in terms of maximal escape rates. These bounds depend on combinatorial information regarding sublevel sets of Green functions. A similar discussion can be found in [DM08, Section 4], where the corresponding results are stated in terms of trees. Thus, we give details for the reader's convenience.

First, we shall briefly recall a few necessary elements from conformal geometry. We say that a Riemann surface A is an *annulus* if its fundamental group $\pi_1(A)$ is isomorphic to $\mathbb{Z}$. For every $z_0 \in \mathbb{C}$ and all $r, R \in \mathbb{R}_{>0}$, with $r < R$,

$$\mathcal{A}_{z_0}(r, R) = \{z \in \mathbb{C} : r < |z - z_0| < R\}$$

is an annulus. In fact, every annulus A is biholomorphic to the punctured disk $\mathbb{D}^*$, the punctured plane $\mathbb{C}^*$ or the round annulus $\mathcal{A}_0(1, R)$ for a unique $R \in (1, +\infty)$. We call *modulus* of an annulus A the number

$$\mathrm{mod}(A) = \begin{cases} \frac{1}{2\pi}\log(R) & \text{if } A \cong \mathcal{A}_0(1, R), \text{ with } R \in (1, +\infty) \\ +\infty & \text{if } A \cong \mathbb{D}^* \text{ or } \mathbb{C}^* \end{cases} .$$

We shall only use the following two facts about moduli of annuli:

- If A is a Riemann surface, B is an annulus and $f\colon A \to B$ is a holomorphic covering map of degree $e \geq 1$, then A is an annulus and
$$\mathrm{mod}(B) = e \cdot \mathrm{mod}(A) .$$
- Given an annulus B, we say that a nonempty connected open subset A of B is a *subannulus* of B if the inclusion $\imath\colon A \hookrightarrow B$ induces an isomorphism $\imath_*\colon \pi_1(A) \to \pi_1(B)$. If B is an annulus and $A_1, \dots, A_r$ are pairwise disjoint subannuli of B, with $r \geq 0$, then
$$\sum_{j=1}^{r} \mathrm{mod}\left(A_j\right) \leq \mathrm{mod}(B) .$$
This statement is known as Grötzsch's inequality.

We refer the reader to [Hub06, Section 3.2] for further details about annuli.

Recall that a compact subset K of $\mathbb{C}$ is full if the set $\mathbb{C} \setminus K$ is connected. Note that, if V is a simply connected open subset of $\mathbb{C}$ and K is a nonempty, connected and full compact subset of V, then $V \setminus K$ is an annulus. Also note that, if $W \subseteq V$ are simply connected open subsets of $\mathbb{C}$ and $K \subseteq L$ are nonempty, connected and full compact subsets of W, then $W \setminus L$ is a subannulus of $V \setminus K$.

We shall crucially use the inequality below, which relates moduli of multipliers at repelling fixed points to moduli of certain annuli.

Lemma 31. *Suppose that U, V are nonempty simply connected open subsets of $\mathbb{C}$ such that $\overline{U}$ is a full compact subset of V and $f\colon U \to V$ is a biholomorphism. Then f has a unique fixed point $z_0 \in U$ and we have*

$$\frac{1}{2\pi} \log \left|f'(z_0)\right| \geq \operatorname{mod}\left(V \setminus \overline{U}\right) .$$

Proof. Set $A = V \setminus \overline{U}$. Note that U, V are biholomorphic to the unit disk $\mathbb{D}$. Now, denote by $g\colon V \to U$ the inverse of f. By the Schwarz lemma, as $U \Subset V$, the map g is a contraction with respect to the Poincaré metric on V. In particular, g has a unique fixed point $z_0 \in U$, which is necessarily attracting for g. Then z_0 is also the unique fixed point for f and z_0 is repelling for f. Now, suppose that $\alpha > \left|f'(z_0)\right|$. Then there exists $R \in \mathbb{R}_{>0}$ such that $\left|f(z) - z_0\right| \leq \alpha \left|z - z_0\right|$ for all $z \in D(z_0, R)$. Since g is contracting with respect to the Poincaré metric on V, there exists $N \geq 0$ such that $g^{\circ N}(V) \subseteq D(z_0, R)$. Take $r \in (0, R)$ such that $D(z_0, r) \subseteq g^{\circ N}(U)$. Note that $g^{\circ N}(A), \dots, g^{\circ n}(A)$ are pairwise disjoint subannuli of $g^{\circ N}(V) \setminus g^{\circ n}\left(\overline{U}\right)$ with modulus $\operatorname{mod}(A)$ for all $n \geq N$. Moreover, for every $n \geq N$, we have

$$D\left(z_0, \frac{r}{\alpha^{n-N}}\right) \subseteq g^{\circ(n-N)}\left(D(z_0, r)\right) \subseteq g^{\circ n}(U)$$

since $\alpha > 1$, and hence $g^{\circ N}(V) \setminus g^{\circ n}\left(\overline{U}\right)$ is a subannulus of $\mathcal{A}_{z_0}\left(\frac{r}{\alpha^{n-N}}, R\right)$. Thus, by Grötzsch's inequality, we have

$$(n - N + 1) \operatorname{mod}(A) \leq \operatorname{mod}\left(\mathcal{A}_{z_0}\left(\frac{r}{\alpha^{n-N}}, R\right)\right) = \frac{1}{2\pi} \log\left(\frac{R}{r}\right) + \frac{n-N}{2\pi} \log(\alpha)$$

for all $n \geq N$, which yields $\operatorname{mod}(A) \leq \frac{1}{2\pi} \log(\alpha)$ by dividing by $n - N$ and letting $n \to +\infty$. Letting $\alpha \to \left|f'(z_0)\right|$, we obtain the desired inequality. Thus, the lemma is proved. $\square$

Now, we shall prove the result below (compare [DM08, Lemma 4.6]). It is a key point in our proof of Theorem B in the complex case.

Lemma 32. *Suppose that $f \in \operatorname{Poly}_d(\mathbb{C})$ has a disconnected filled Julia set $\mathcal{K}_f$, $\eta \in [M_f, +\infty)$, $U_0, \dots, U_{p-1}$ are (not necessarily distinct) connected components of $\left\{g_f < \frac{\eta}{d^k}\right\}$, with $k \geq 0$ and $p \geq 1$, $V_0, \dots, V_{p-1}$ are the connected components of $\left\{g_f < \frac{\eta}{d^{k-1}}\right\}$ containing $U_0, \dots, U_{p-1}$, respectively, and f induces a biholomorphism from U_j to $V_{j+1 \pmod p}$ for all $j \in \{0, \dots, p-1\}$. Then $f^{\circ p}$ has a unique fixed point $z_0 \in \mathbb{C}$ such that $f^{\circ j}(z_0) \in U_j$ for all $j \in \{0, \dots, p-1\}$. Furthermore, we have*

$$\log\left|\left(f^{\circ p}\right)'(z_0)\right| \geq (d-1) \left(\sum_{j=0}^{p-1} \frac{1}{d_j}\right) \eta \,,$$

where d_j denotes the degree of $f^{\circ k}\colon V_j \to \{g_f < d \cdot \eta\}$ for all $j \in \{0, \dots, p-1\}$.

Proof. Denote by g_j the inverse of $f\colon U_j \to V_{j+1 \pmod p}$ for $j \in \{0, \dots, p-1\}$, and define

$$W = g_0 \circ \dots \circ g_{p-1}(V_0) \,.$$

Then W, V_0 are simply connected open subsets of $\mathbb{C}$ such that $\overline{W}$ is a full compact subset of V_0 and $f^{\circ p}\colon W \to V_0$ is a biholomorphism. Therefore, by Lemma 31, the map $f^{\circ p}$ has a unique fixed point $z_0 \in W$ and we have

$$\log\left|\left(f^{\circ p}\right)'(z_0)\right| \geq 2\pi \operatorname{mod}\left(V_0 \setminus \overline{W}\right) .$$

Observe that z_0 is the unique fixed point for $f^{\circ p}$ that satisfies $f^{\circ j}(z_0) \in U_j$ for all $j \in \{0, \dots, p-1\}$. Thus, it remains to prove the desired inequality. Note that the $g_0 \circ \dots \circ g_{j-1}\left(V_j \setminus \overline{U_j}\right)$, with $j \in \{0, \dots, p-1\}$, are pairwise disjoint subannuli of $V_0 \setminus \overline{W}$. Therefore, by Grötzsch's inequality, we have

$$\log\left|\left(f^{\circ p}\right)'(z_0)\right| \geq 2\pi \sum_{j=0}^{p-1} \operatorname{mod}\left(V_j \setminus \overline{U_j}\right) .$$

Suppose that $j \in \{0, \dots, p-1\}$, and let us show that $\operatorname{mod}\left(V_j \setminus \overline{U_j}\right) \geq \frac{1}{2\pi}\left(\frac{d-1}{d_j}\right)\eta$. Define $\eta_j^{(0)} < \dots < \eta_j^{(N_j)}$, with $N_j \geq 1$, by $\eta_j^{(0)} = \frac{\eta}{d^k}$ and $\eta_j^{(N_j)} = \frac{\eta}{d^{k-1}}$ and

$$\left\{\eta_j^{(1)}, \dots, \eta_j^{(N_j-1)}\right\} = \left\{g_f(\gamma) : \gamma \in V_j,\, g_f(\gamma) > \frac{\eta}{d^k},\, \left(f^{\circ k}\right)'(\gamma) = 0\right\} .$$

For $\ell \in \{1, \dots, N_j\}$, denote by $D_j^{(\ell)}$ the connected component of $\left\{g_f < \eta_j^{(\ell)}\right\}$ that contains U_j and define

$$A_j^{(\ell)} = D_j^{(\ell)} \setminus \left\{g_f \leq \eta_j^{(\ell-1)}\right\} .$$

Then $f^{\circ k}$ induces a covering map from $A_j^{(\ell)}$ to $\left\{d^k \eta_j^{(\ell-1)} < g_f < d^k \eta_j^{(\ell)}\right\}$ of degree at most d_j for each $\ell \in \{1, \dots, N_j\}$. Moreover, $A_j^{(1)}, \dots, A_j^{(N_j)}$ are pairwise disjoint subannuli of $V_j \setminus \overline{U_j}$. Therefore, we have

$$\operatorname{mod}\left(V_j \setminus \overline{U_j}\right) \geq \frac{1}{d_j} \sum_{\ell=1}^{N_j} \operatorname{mod}\left(\left\{d^k \eta_j^{(\ell-1)} < g_f < d^k \eta_j^{(\ell)}\right\}\right)$$

Finally, denote by $\phi_f\colon \{g_f > M_f\} \to \mathbb{C} \setminus \overline{D\left(0, \exp\left(M_f\right)\right)}$ a Böttcher coordinate of f at infinity, which satisfies $g_f = \log|\phi_f|$. Then ϕ_f induces a biholomorphism from $\left\{d^k \eta_j^{(\ell-1)} < g_f < \eta_j^{(\ell)}\right\}$ onto the round annulus $\mathcal{A}_0\left(\exp\left(d^k \eta_j^{(\ell-1)}\right), \exp\left(d^k \eta_j^{(\ell)}\right)\right)$ for each $\ell \in \{1, \dots, N_j\}$. Therefore, we have

$$\operatorname{mod}\left(\left\{d^k \eta_j^{(\ell-1)} < g_f < d^k \eta_j^{(\ell)}\right\}\right) = \frac{d^k}{2\pi}\left(\eta_j^{(\ell)} - \eta_j^{(\ell-1)}\right)$$

for all $\ell \in \{1, \dots, N_j\}$, and hence

$$\operatorname{mod}\left(V_j \setminus \overline{U_j}\right) \geq \frac{d^k}{2\pi d_j}\left(\eta_j^{(N_j)} - \eta_j^{(0)}\right) = \frac{1}{2\pi}\left(\frac{d-1}{d_j}\right)\eta .$$

This completes the proof of the lemma. □

Remark 33. In order to prove Theorem B in the Archimedean case, we shall only apply Lemma 32 with $\eta = M_f$, $k \in \{0, 1\}$ and $p \in \{1, 2\}$. Nonetheless, our general statement of Lemma 32 also allows us to prove a slightly weaker version of [EL92, Theorem 1.6], which provides a lower bound on the characteristic exponent at any periodic point in terms of the minimum of the Green function on the set of critical points (see Proposition 83).

3.4. **Proof of Theorem B in the Archimedean case.** Finally, let us combine Lemmas 25 and 32 in order to prove Theorem B for polynomial maps defined over an algebraically closed Archimedean valued field.

Proof of Theorem B in the Archimedean case. Suppose that K is an algebraically closed field equipped with an Archimedean absolute value $|.|_\infty$ and $f \in \mathrm{Poly}_d(K)$. Define $\widehat{K}$ to be the completion of K. Then, by Ostrowski's theorem, there exist an embedding $\sigma\colon \widehat{K} \hookrightarrow \mathbb{C}$ and $s \in (0,1]$ such that $|z|_\infty = |\sigma(z)|^s$ for all $z \in \widehat{K}$, where $|.|$ denotes the usual absolute value on $\mathbb{C}$ (see [Neu99, Chapter II, (4.2)]). Since K is algebraically closed, the critical points and the periodic points for $\sigma(f)$ in $\mathbb{C}$ all lie in $\sigma(K)$. Therefore, $M_f = s \cdot M_{\sigma(f)}$ and $M_f^{(p)} = s \cdot M_{\sigma(f)}^{(p)}$ for each $p \geq 1$. Thus, replacing f by $\sigma(f)$ if necessary, we may assume that $f \in \mathrm{Poly}_d(\mathbb{C})$.

First, assume that $\mathcal{K}_f$ is connected. Then $M_f = 0$. Let us prove that $M_f^{(1)} \geq 0$. Denote by $\lambda_1, \dots, \lambda_d$ the multipliers of f at its fixed points repeated according to their multiplicities. Then either $\lambda_j = 1$ for some $j \in \{1, \dots, d\}$ or $\sum\limits_{j=1}^{d} \frac{1}{1-\lambda_j} = 0$ by the holomorphic fixed-point formula. Note that, if $\lambda \in D(0,1)$, then $\Re\left(\frac{1}{1-\lambda}\right) > \frac{1}{2}$. Therefore, there exists $j \in \{1, \dots, d\}$ such that $|\lambda_j| \geq 1$, and thus $M_f^{(1)} \geq 0$.

Thus, assume now that $\mathcal{K}_f$ is disconnected. Denote by $V_1, \dots, V_N$, with $N \geq 2$, the connected components of $\{g_f < M_f\}$. For $j \in \{1, \dots, N\}$, denote by $d_j \geq 1$ the degree of $f\colon V_j \to \{g_f < d \cdot M_f\}$. Then $\sum\limits_{j=1}^{N} d_j = d$. Let us consider three cases.

Suppose that $d_j = 1$ for some $j \in \{1, \dots, N\}$. Note that this holds if $d \in \{2, 3\}$. Then f induces a biholomorphism from V_j to $\{g_f < d \cdot M_f\}$. Thus, by Lemma 32, the map f has a unique fixed point $z_0 \in V_j$ and we have $\log|f'(z_0)| \geq (d-1)M_f$. In particular, $M_f^{(1)} \geq (d-1)M_f$.

Now, suppose that the condition (1) of Lemma 25 is satisfied and $d_j \geq 2$ for all $j \in \{1, \dots, N\}$. Then there exist $j \in \{1, \dots, N\}$ and a connected component U_j of $\left\{g_f < \frac{M_f}{d}\right\}$ contained in V_j such that f induces a biholomorphism from U_j to V_j. Therefore, by Lemma 32, the map f has a unique fixed point $z_0 \in U_j$ and we have $\log|f'(z_0)| \geq \frac{d-1}{d_j} M_f$. Moreover, $d_j = d - \sum\limits_{k \neq j} d_k \leq d - 2$. Thus, $M_f^{(1)} \geq \frac{d-1}{d-2} M_f$.

Finally, suppose that the condition (2) of Lemma 25 is satisfied. Then there are connected components U_1, U_2 of $\left\{g_f < \frac{M_f}{d}\right\}$ contained in V_1, V_2, respectively, such that f induces biholomorphisms from U_1 to V_2 and from U_2 to V_1. By Lemma 32, it follows that the map $f^{\circ 2}$ has a unique fixed point $z_0 \in \mathbb{C}$ such that $z_0 \in U_1$ and $f(z_0) \in U_2$ and we have

$$\log\left|\left(f^{\circ 2}\right)'(z_0)\right| \geq (d-1)\left(\frac{1}{d_1} + \frac{1}{d_2}\right) M_f \geq (d-1)\left(\frac{1}{d_1} + \frac{1}{d-d_1}\right) M_f \,.$$

Therefore, $M_f^{(2)} \geq C_d \cdot M_f$, where

$$C_d = \min_{j \in \{1, \dots, d-1\}} \frac{d-1}{2}\left(\frac{1}{j} + \frac{1}{d-j}\right) = \begin{cases} \frac{2(d-1)}{d} & \text{if } d \text{ is even} \\ \frac{2d}{d+1} & \text{if } d \text{ is odd} \end{cases} .$$

This completes the proof of the theorem. ∎

Remark 34. Using Remark 27, one can easily adapt the proof above to show that $M_f^{(2)} \geq M_f$ for all $f \in \mathrm{Poly}_d(\mathbb{C})$ with disconnected filled Julia set $\mathcal{K}_f$.

Finally, note that Theorem A is proved since it is an immediate consequence of Theorem B in the Archimedean case.

3.5. **Consequences of Theorem A.** To conclude this section, let us apply here Theorem A to establish results about the morphism $\mathrm{Mult}_d^{(2)}$. We shall also deduce Corollaries A.1 and A.2. For brevity, we omit the analogous results concerning the multipliers at the fixed points alone when $d \in \{2,3\}$, as these can be obtained in a completely similar way.

To conclude this section, let us use here Theorem 1.2 to obtain results about the morphism $\mathrm{Mult}_d^{(2)}$. We shall also deduce Corollaries A.1 and A.2. For brevity, we omit the analogous results concerning the multipliers at the fixed points alone when $d \in \{2,3\}$, as these can be obtained in a completely similar way.

First, we have the following direct consequence of Theorem A:

Corollary 35. *The holomorphic map* $\mathrm{Mult}_d^{(2)} \colon \mathcal{P}_d(\mathbb{C}) \to \mathbb{C}^d \times \mathbb{C}^{\frac{d(d-1)}{2}}$ *is proper.*

Proof. Note that, for every $f \in \mathrm{Poly}_d(\mathbb{C})$ and every $p \geq 1$, we have

$$\exp\left(p \cdot M_f^{(p)}\right) = \max_{\lambda \in \Lambda_f^{(p)}} |\lambda| \leq 1 + \max_{j \in \left\{1,\dots,N_d^{(p)}\right\}} \left|\sigma_{d,j}^{(p)}([f])\right|$$

by Cauchy's bound on roots of a complex polynomial. As a result, by Theorem A, if $(f_n)_{n\geq 0}$ is any sequence of elements of $\mathrm{Poly}_d(\mathbb{C})$ that degenerates in $\mathcal{P}_d(\mathbb{C})$, then $\lim\limits_{n\to+\infty} \mathrm{Mult}_d^{(2)}([f_n]) = \infty$ in $\mathbb{C}^d \times \mathbb{C}^{\frac{d(d-1)}{2}}$. Thus, the corollary is proved. □

To derive further consequences of Theorem A, we shall apply the general result below from algebraic geometry.

Lemma 36. *Suppose that X, Y are two affine varieties over $\mathbb{Q}$ and $\Psi \colon X \to Y$ is a morphism such that the induced holomorphic map $\Psi \colon X(\mathbb{C}) \to Y(\mathbb{C})$ is proper. Then Ψ is a finite morphism.*

Proof. Denote here by $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ the base changes of X and Y to $\mathbb{C}$, respectively. As the holomorphic map $\Psi \colon X(\mathbb{C}) \to Y(\mathbb{C})$ is proper, the morphism $\Psi \colon X_{\mathbb{C}} \to Y_{\mathbb{C}}$ is proper (see [SGA71, Exposé XII, Proposition 3.2]). Therefore, since $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ are affine, $\Psi \colon X_{\mathbb{C}} \to Y_{\mathbb{C}}$ is a finite morphism (see [Liu02, Chapter 3, Lemma 3.17]). As a result, the morphism $\Psi \colon X \to Y$ is finite (see [Gro65, (2.7.1)]). □

Combining Corollary 35 and Lemma 36, we immediately obtain the following:

Corollary 37. *The morphism* $\mathrm{Mult}_d^{(2)} \colon \mathcal{P}_d \to \mathbb{A}^d \times \mathbb{A}^{\frac{d(d-1)}{2}}$ *is finite.*

We shall now prove Corollaries A.1 and A.2. To do this, we first present general results about polynomial maps with rational coefficients that allow one to deduce facts over arbitrary valued fields from analogous facts in the complex setting.

Given a valued field K, we denote here by $|.|_K$ its absolute value and, for $n \geq 1$, we denote by $\|.\|_{K^n}$ the norm on K^n defined by

$$\|\boldsymbol{t}\|_{K^n} = \max_{j \in \{1,\dots,n\}} |t_j|_K \quad \text{for} \quad \boldsymbol{t} = (t_1,\dots,t_n) \in K^n .$$

Given a valued field K and an integer $n \geq 1$, we also define

$$(n)_K = \begin{cases} n & \text{if } K \text{ is Archimedean} \\ 1 & \text{if } K \text{ is non-Archimedean} \end{cases},$$

so that

$$\forall \boldsymbol{t} \in K^n,\ \left| \sum_{j=1}^{n} t_j \right|_K \leq (n)_K \|\boldsymbol{t}\|_{K^n}$$

by the triangle inequality.

Lemma 38. *Suppose that $\boldsymbol{F} \colon \mathbb{A}^m \to \mathbb{A}^n$, with $m, n \geq 1$, is a morphism such that the induced holomorphic map $\boldsymbol{F} \colon \mathbb{C}^m \to \mathbb{C}^n$ has no zero in $\mathbb{C}^m$. Then, for every valued field K of characteristic 0, there exist some $\alpha_K^{(0)} \in \mathbb{R}_{>0}$ depending only on the restriction of $|.|_K$ to $\mathbb{Q}$ and some $\delta \in \mathbb{R}_{\geq 0}$ not depending on K such that*

$$\forall \boldsymbol{t} \in K^m,\ \|\boldsymbol{F}(\boldsymbol{t})\|_{K^n} \geq \alpha_K^{(0)} \max\{1, \|\boldsymbol{t}\|_{K^m}\}^{-\delta}.$$

Moreover, we can take $\alpha_K^{(0)} = 1$ for every non-Archimedean field K with characteristic 0 and residue characteristic outside a finite set $S^{(0)}$ of prime numbers.

Proof. Write $\boldsymbol{F} = (F_1, \ldots, F_n)$, with $F_1, \ldots, F_n \in \mathbb{Q}[T_1, \ldots, T_m]$. Since $F_1, \ldots, F_n$ have no common zero in $\mathbb{C}^m$ by hypothesis, it follows from the Nullstellensatz that there exist $G_1, \ldots, G_n \in \mathbb{Q}[T_1, \ldots, T_m]$ such that

$$\sum_{j=1}^{n} F_j(T_1, \ldots, T_m)\, G_j(T_1, \ldots, T_m) = 1.$$

Define $\delta = \max\limits_{j \in \{1, \ldots, n\}} \deg(G_j)$. For $j \in \{1, \ldots, n\}$, write

$$G_j(T_1, \ldots, T_m) = \sum_{\boldsymbol{\ell} \in \{0, \ldots, \delta\}^m} a_{j,\boldsymbol{\ell}} \prod_{k=1}^{m} T_k^{\ell_k} \in \mathbb{Q}[T_1, \ldots, T_m].$$

Consider the finite set

$$S^{(0)} = \left\{ p \text{ prime} : \max_{\substack{j \in \{1, \ldots, n\} \\ \boldsymbol{\ell} \in \{0, \ldots, \delta\}^m}} |a_{j,\boldsymbol{\ell}}|_p \neq 1 \right\},$$

where $|.|_p$ denotes the p-adic absolute value on $\mathbb{Q}$ for each prime number p. Now, suppose that K is a valued field of characteristic 0. Then we have

$$1 \leq (n)_K \|\boldsymbol{F}(\boldsymbol{t})\|_{K^n} \left(\max_{j \in \{1, \ldots, n\}} |G_j(\boldsymbol{t})|_K \right) \leq A_K^{(0)} \|\boldsymbol{F}(\boldsymbol{t})\|_{K^n} \max\{1, \|\boldsymbol{t}\|_{K^m}\}^{\delta}$$

for all $\boldsymbol{t} \in K^m$ by the triangle inequality, where

$$A_K^{(0)} = (n)_K ((\delta + 1)^m)_K \left(\max_{\substack{j \in \{1, \ldots, n\} \\ \boldsymbol{\ell} \in \{0, \ldots, \delta\}^m}} |a_{j,\boldsymbol{\ell}}|_K \right) \in \mathbb{R}_{>0}.$$

Therefore, setting $\alpha_K^{(0)} = \left(A_K^{(0)} \right)^{-1}$, we have

$$\forall \boldsymbol{t} \in K^m,\ \|\boldsymbol{F}(\boldsymbol{t})\|_{K^n} \geq \alpha_K^{(0)} \max\{1, \|\boldsymbol{t}\|_{K^m}\}^{-\delta}.$$

Finally, note that $\alpha_K^{(0)}$ depends only on the restriction of $|.|_K$ to $\mathbb{Q}$. Furthermore, we have $\alpha_K^{(0)} = 1$ if K is non-Archimedean with residue characteristic not in $S^{(0)}$. This completes the proof of the lemma. $\square$

Also, we have the following result about the growth of proper polynomial maps at infinity:

Lemma 39. *Suppose that $\boldsymbol{F}\colon \mathbb{A}^m \to \mathbb{A}^n$, with $m, n \geq 1$, is a morphism such that the induced holomorphic map $\boldsymbol{F}\colon \mathbb{C}^m \to \mathbb{C}^n$ is proper. Then, for every valued field K of characteristic 0, there exist some $\alpha_K^{(\infty)}, R_K^{(\infty)} \in \mathbb{R}_{>0}$ depending only on the restriction of $|.|_K$ to $\mathbb{Q}$ and some $\beta \in \mathbb{R}_{>0}$ not depending on K such that*

$$\forall \boldsymbol{t} \in K^m,\ \|\boldsymbol{t}\|_{K^m} > R_K^{(\infty)} \Longrightarrow \|\boldsymbol{F}(\boldsymbol{t})\|_{K^n} \geq \alpha_K^{(\infty)} \|\boldsymbol{t}\|_{K^m}^{\beta} .$$

Moreover, we can take $\alpha_K^{(\infty)} = 1$ and $R_K^{(\infty)} = 1$ for every non-Archimedean field K with characteristic 0 and residue characteristic outside a finite set $S^{(\infty)}$ of prime numbers.

Proof. Write $\boldsymbol{F} = (F_1, \ldots, F_n)$, with $F_1, \ldots, F_n \in \mathbb{Q}[T_1, \ldots, T_m]$. As the holomorphic map $\boldsymbol{F}\colon \mathbb{C}^m \to \mathbb{C}^n$ is proper by hypothesis, $\boldsymbol{F}\colon \mathbb{A}^m \to \mathbb{A}^n$ is a finite morphism by Lemma 36, and hence T_j is integral over $\mathbb{Q}[F_1, \ldots, F_n]$ for each $j \in \{1, \ldots, m\}$. Thus, for each $j \in \{1, \ldots, m\}$, there exist $P_{j,0}, \ldots, P_{j,D_j-1} \in \mathbb{Q}[X_1, \ldots, X_n]$, with $D_j \geq 1$, such that

$$T_j^{D_j} = \sum_{k=0}^{D_j-1} P_{j,k}\left(F_1\left(T_1, \ldots, T_m\right), \ldots, F_n\left(T_1, \ldots, T_m\right)\right) T_j^k .$$

Define

$$\gamma = \max_{\substack{j \in \{1, \ldots, m\} \\ k \in \{0, \ldots, D_j-1\}}} \deg\left(P_{j,k}\right) \in \mathbb{Z}_{\geq 1} \quad \text{and} \quad \beta = \frac{1}{\gamma} \in \mathbb{R}_{>0} .$$

For $j \in \{1, \ldots, m\}$ and $k \in \{0, \ldots, D_j - 1\}$, write

$$P_{j,k}\left(X_1, \ldots, X_n\right) = \sum_{\boldsymbol{\ell} \in \{0, \ldots, \gamma\}^n} b_{j,k,\boldsymbol{\ell}} \prod_{r=1}^{n} X_r^{\ell_r} \in \mathbb{Q}\left[X_1, \ldots, X_n\right] .$$

Consider the finite set

$$S^{(\infty)} = \left\{ p \text{ prime} : \max_{\substack{j \in \{1, \ldots, m\} \\ k \in \{0, \ldots, D_j-1\} \\ \boldsymbol{\ell} \in \{0, \ldots, \gamma\}^n}} \left|b_{j,k,\boldsymbol{\ell}}\right|_p \neq 1 \right\} ,$$

where $|.|_p$ denotes the p-adic absolute value on $\mathbb{Q}$ for each prime number p. Now, suppose that K is a valued field of characteristic 0. Set

$$A_K^{(\infty)} = \left(\max_{j \in \{1, \ldots, m\}} \left(D_j\right)_K \right) \left((\gamma+1)^n\right)_K \left(\max_{\substack{j \in \{1, \ldots, m\} \\ k \in \{0, \ldots, D_j-1\} \\ \boldsymbol{\ell} \in \{0, \ldots, \gamma\}^n}} \left|b_{j,k,\boldsymbol{\ell}}\right|_K \right) \in \mathbb{R}_{>0} .$$

For every $\boldsymbol{t} \in K^m$, as $t_j^{D_j} = \sum_{k=0}^{D_j-1} P_{j,k}\left(\boldsymbol{F}(\boldsymbol{t})\right) t_j^k$ for all $j \in \{1, \dots, m\}$, we have

$$\forall j \in \{1, \dots, m\},\ |t_j|_K^{D_j} \leq A_K^{(\infty)} \max\left\{1, \|\boldsymbol{F}(\boldsymbol{t})\|_{K^n}\right\}^\gamma \max\left\{1, |t_j|_K\right\}^{D_j-1}$$

by the triangle inequality. Therefore, we have

$$\forall \boldsymbol{t} \in K^m,\ \|\boldsymbol{t}\|_{K^m} \geq 1 \Longrightarrow \|\boldsymbol{t}\|_{K^m} \leq A_K^{(\infty)} \max\left\{1, \|\boldsymbol{F}(\boldsymbol{t})\|_{K^n}\right\}^\gamma .$$

As a result, setting $\alpha_K^{(\infty)} = \left(A_K^{(\infty)}\right)^{-\beta}$ and $R_K^{(\infty)} = \max\left\{1, A_K^{(\infty)}\right\}$, we have

$$\forall \boldsymbol{t} \in K^m,\ \|\boldsymbol{t}\|_{K^m} > R_K^{(\infty)} \Longrightarrow \|\boldsymbol{F}(\boldsymbol{t})\|_{K^n} \geq \alpha_K^{(\infty)} \|\boldsymbol{t}\|_{K^m}^\beta .$$

Finally, note that both $\alpha_K^{(\infty)}$ and $R_K^{(\infty)}$ depend only on the restriction of $|.|_K$ to $\mathbb{Q}$. Moreover, we have $\alpha_K^{(\infty)} = 1$ and $R_K^{(\infty)} = 1$ if K is non-Archimedean with residue characteristic not in $S^{(\infty)}$. This completes the proof of the lemma. □

Remark 40. Given a polynomial map $\boldsymbol{F}\colon \mathbb{C}^m \to \mathbb{C}^n$, with $m, n \geq 1$, the supremum of all $\beta \in \mathbb{R}$ for which there exist $\alpha, R \in \mathbb{R}_{>0}$ such that $\|\boldsymbol{F}(\boldsymbol{t})\|_{\mathbb{C}^n} \geq \alpha \|\boldsymbol{t}\|_{\mathbb{C}^m}^\beta$ for all $\boldsymbol{t} \in \mathbb{C}^m$ such that $\|\boldsymbol{t}\|_{\mathbb{C}^m} > R$ is known as the Łojasiewicz exponent of $\boldsymbol{F}$ at infinity. We refer the reader to [CK97, Corollary 2] for an analytic proof that every proper complex polynomial map has positive Łojasiewicz exponent at infinity.

Combining Lemmas 38 and 39, we easily obtain the general result below.

Lemma 41. *Suppose that $\boldsymbol{F}\colon \mathbb{A}^m \to \mathbb{A}^n$, with $m, n \geq 1$, is a morphism such that the induced holomorphic map $\boldsymbol{F}\colon \mathbb{C}^m \to \mathbb{C}^n$ is proper and has no zero in $\mathbb{C}^m$. Then, for every valued field K of characteristic 0, there exist some $\alpha_K \in \mathbb{R}_{>0}$ depending only on the restriction of $|.|_K$ to $\mathbb{Q}$ and some $\beta \in \mathbb{R}_{>0}$ not depending on K such that*

$$\forall \boldsymbol{t} \in K^m,\ \|\boldsymbol{F}(\boldsymbol{t})\|_{K^n} \geq \alpha_K \max\left\{1, \|\boldsymbol{t}\|_{K^m}\right\}^\beta .$$

Moreover, we can take $\alpha_K = 1$ for every non-Archimedean field K with characteristic 0 and residue characteristic outside a finite set S of prime numbers.

Proof. Suppose that K is any valued field of characteristic 0. Define

$$\alpha_K = \min\left\{\alpha_K^{(0)} \max\left\{1, R_K^{(\infty)}\right\}^{-\beta-\delta}, \alpha_K^{(\infty)}\right\} \in \mathbb{R}_{>0} \quad \text{and} \quad S = S^{(0)} \cup S^{(\infty)} ,$$

with $\alpha_K^{(0)} \in \mathbb{R}_{>0}$, $\delta \in \mathbb{R}_{\geq 0}$, $S^{(0)}$ as in Lemma 38 and $\alpha_K^{(\infty)}, R_K^{(\infty)}, \beta \in \mathbb{R}_{>0}$, $S^{(\infty)}$ as in Lemma 39. Then α_K depends only on the restriction of $|.|_K$ to $\mathbb{Q}$, and we have $\alpha_K = 1$ if K is non-Archimedean with residue characteristic outside S. Moreover, for every $\boldsymbol{t} \in K^m$ such that $\|\boldsymbol{t}\|_{K^m} \leq \max\left\{1, R_K^{(\infty)}\right\}$, we have

$$\|\boldsymbol{F}(\boldsymbol{t})\|_{K^n} \geq \alpha_K^{(0)} \max\left\{1, R_K^{(\infty)}\right\}^{-\delta} \geq \alpha_K \max\left\{1, \|\boldsymbol{t}\|_{K^m}\right\}^\beta .$$

Also, for every $\boldsymbol{t} \in K^m$ such that $\|\boldsymbol{t}\|_{K^m} > \max\left\{1, R_K^{(\infty)}\right\}$, we clearly have

$$\|\boldsymbol{F}(\boldsymbol{t})\|_{K^n} \geq \alpha_K \max\left\{1, \|\boldsymbol{t}\|_{K^m}\right\}^\beta .$$

Thus, the lemma is proved. □

We shall now apply the lemma above to obtain results about the multipliers of polynomials over arbitrary algebraically closed valued fields of characteristic 0. To do this, we work with polynomial maps in a particular form, which was introduced by Ingram in [Ing12]. Although the two claims below concerning this normal form are already known, we include proofs for the reader's convenience.

Given any field K of characteristic 0 and $\boldsymbol{c} = (c_1, \dots, c_{d-1}) \in K^{d-1}$, define

$$f_{\boldsymbol{c}}(z) = \frac{1}{d} z^d + \sum_{j=1}^{d-1} \frac{(-1)^j \tau_j(\boldsymbol{c})}{d-j} z^{d-j} \in \mathrm{Poly}_d(K)\,,$$

where $\tau_1(\boldsymbol{c}), \dots, \tau_{d-1}(\boldsymbol{c})$ denote the elementary symmetric functions of $c_1, \dots, c_{d-1}$, so that

$$f_{\boldsymbol{c}}(0) = 0 \quad \text{and} \quad f_{\boldsymbol{c}}'(z) = \prod_{j=1}^{d-1} (z - c_j)\ .$$

Consider the morphism $F\colon \mathbb{A}^{d-1} \to \mathcal{P}_d$ defined by $F(\boldsymbol{c}) = f_{\boldsymbol{c}}$.

Claim 42. The holomorphic map $F\colon \mathbb{C}^{d-1} \to \mathcal{P}_d(\mathbb{C})$ is proper.

Proof. Assume that $(\boldsymbol{c}_n)_{n\geq 0}$ is a sequence of elements of $\mathbb{C}^{d-1}$ such that $([f_{\boldsymbol{c}_n}])_{n\geq 0}$ converges in $\mathcal{P}_d(\mathbb{C})$. We shall show that $(\boldsymbol{c}_n)_{n\geq 0}$ is bounded in $\mathbb{C}^{d-1}$. There exists a sequence $(\phi_n)_{n\geq 0}$ of elements of $\mathrm{Aff}(\mathbb{C})$ such that the sequence $(f_n)_{n\geq 0}$ given by $f_n = \phi_n \centerdot f_{\boldsymbol{c}_n}$ converges to some $g \in \mathrm{Poly}_d(\mathbb{C})$. The multiset of all the fixed points for f_n tends to the multiset of all the fixed points for g in $\mathbb{C}^d/\mathfrak{S}_d$ as $n \to +\infty$. In particular, $(\phi_n(0))_{n\geq 0}$ is bounded in $\mathbb{C}$. In addition, the multiset of all the critical points for f_n tends to the multiset of all the critical points for g in $\mathbb{C}^{d-1}/\mathfrak{S}_{d-1}$ as $n \to +\infty$. As a result, writing $\boldsymbol{c}_n = \left(c_n^{(1)}, \dots, c_n^{(d-1)}\right)$ for each $n \geq 0$, the sequence $\left(\phi_n\left(c_n^{(j)}\right)\right)_{n\geq 0}$ is bounded in $\mathbb{C}$ for each $j \in \{1, \dots, d-1\}$. Furthermore, writing $\phi_n(z) = \alpha_n z + \beta_n$ for all $n \geq 0$, the polynomial f_n has leading coefficient $\frac{\alpha_n^{1-d}}{d}$ for all $n \geq 0$, which yields $\lim\limits_{n\to+\infty} |\alpha_n| = |d \cdot a_d|^{\frac{-1}{d-1}} \in \mathbb{R}_{>0}$, where $a_d \in \mathbb{C}^*$ denotes the leading coefficient of g. Therefore, as $c_n^{(j)} = \frac{\phi_n\left(c_n^{(j)}\right) - \phi_n(0)}{\alpha_n}$ for all $j \in \{1, \dots, d-1\}$ and all $n \geq 0$, the sequence $(\boldsymbol{c}_n)_{n\geq 0}$ is bounded in $\mathbb{C}^{d-1}$. This completes the proof of the claim. □

Moreover, using the triangle inequality, we easily obtain an upper bound on the Green functions of the polynomials $f_{\boldsymbol{c}}$, with K an algebraically closed valued field of characteristic 0 and $\boldsymbol{c} \in K^{d-1}$.

Claim 43. For each algebraically closed valued field K of characteristic 0, we have

$$g_{f_{\boldsymbol{c}}}(z) \leq \log^+\left(\max\left\{\|\boldsymbol{c}\|_{K^{d-1}}, |z|_K\right\}\right) + \Delta_K$$

for all $\boldsymbol{c} \in K^{d-1}$ and all $z \in K$, and in particular $M_{f_{\boldsymbol{c}}} \leq \log^+\|\boldsymbol{c}\|_{K^{d-1}} + \Delta_K$ for all $\boldsymbol{c} \in K^{d-1}$, where

$$\Delta_K = \frac{1}{d-1}\log(d)_K + \frac{1}{d-1}\log\left(\max_{j\in\{0,\dots,d-1\}} \frac{1}{|d-j|_K}\left(\binom{d-1}{j}\right)_K\right) \in \mathbb{R}_{\geq 0}\,.$$

Proof. For every $\boldsymbol{c} \in K^{d-1}$ and every $z \in K$, we have

$$|f_{\boldsymbol{c}}(z)|_K \leq (d)_K \left(\max_{j \in \{0,\dots,d-1\}} \left| \frac{(-1)^j \tau_j(\boldsymbol{c})}{d-j} z^{d-j} \right|_K \right) \leq \delta_K \max\left\{ \|\boldsymbol{c}\|_{K^{d-1}}, |z|_K \right\}^d$$

by the triangle inequality, where $\tau_0(\boldsymbol{c}) = 1$ by convention and

$$\delta_K = (d)_K \left(\max_{j \in \{0,\dots,d-1\}} \frac{1}{|d-j|_K} \left(\binom{d-1}{j} \right)_K \right) \in \mathbb{R}_{\geq 1} .$$

It follows by induction that

$$|f_{\boldsymbol{c}}^{\circ n}(z)|_K \leq \delta_K^{\frac{d^n-1}{d-1}} \max\left\{ 1, \|\boldsymbol{c}\|_{K^{d-1}}, |z|_K \right\}^{d^n}$$

for all $\boldsymbol{c} \in K^{d-1}$, all $z \in K$ and all $n \geq 0$. Therefore, for every $\boldsymbol{c} \in K^{d-1}$ and every $z \in K$, we have

$$\frac{1}{d^n} \log^+ |f_{\boldsymbol{c}}^{\circ n}(z)|_K \leq \frac{d^n - 1}{d^n(d-1)} \log(\delta_K) + \log^+\left(\max\left\{ \|\boldsymbol{c}\|_{K^{d-1}}, |z|_K \right\} \right)$$

for all $n \geq 0$, which yields the desired result by letting $n \to +\infty$. Thus, the claim is proved. □

Finally, we obtain the following result regarding polynomial maps over arbitrary algebraically closed valued fields of characteristic 0:

Proposition 44. *Assume that K is an algebraically closed valued field of characteristic 0. Then there exist some $A \in \mathbb{R}_{>0}$ not depending on K and some $B_K \in \mathbb{R}$ depending only on the restriction of $|.|_K$ to $\mathbb{Q}$ such that*

$$\max\left\{ M_f^{(1)}, M_f^{(2)} \right\} \geq A \cdot M_f + B_K$$

for all $f \in \mathrm{Poly}_d(K)$. Furthermore, we can take $B_K = 0$ if K is non-Archimedean with residue characteristic outside some finite set S of prime numbers.

Proof. Consider the morphism $\boldsymbol{G} \colon \mathbb{A}^{d-1} \to \mathbb{A}^d \times \mathbb{A}^{\frac{d(d-1)}{2}}$ defined by

$$\boldsymbol{G}(\boldsymbol{c}) = \mathrm{Mult}_d^{(2)}([f_{\boldsymbol{c}}]) = \left(\left(\sigma_{d,j}^{(1)}([f_{\boldsymbol{c}}]) \right)_{1 \leq j \leq d}, \left(\sigma_{d,j}^{(2)}([f_{\boldsymbol{c}}]) \right)_{1 \leq j \leq \frac{d(d-1)}{2}} \right) .$$

Then the holomorphic map $\boldsymbol{G} \colon \mathbb{C}^{d-1} \to \mathbb{C}^d \times \mathbb{C}^{\frac{d(d-1)}{2}}$ has no zero in $\mathbb{C}^{d-1}$ because $d + \sum_{j=1}^{d} (-1)^j (d-j) \sigma_{d,j}^{(1)} = 0$ by the holomorphic fixed-point formula. In addition, the map $\boldsymbol{G} \colon \mathbb{C}^{d-1} \to \mathbb{C}^d \times \mathbb{C}^{\frac{d(d-1)}{2}}$ is proper by Corollary 35 and Claim 42. Thus, by Lemma 41, there exist some $A' \in \mathbb{R}_{>0}$ not depending on K and some $B'_K \in \mathbb{R}$ depending only on the restriction of $|.|_K$ to $\mathbb{Q}$ such that

$$\forall \boldsymbol{c} \in K^{d-1},\ \log \|\boldsymbol{G}(\boldsymbol{c})\|_{K^d \times K^{\frac{d(d-1)}{2}}} \geq A' \cdot \log^+ \|\boldsymbol{c}\|_{K^{d-1}} + B'_K .$$

Moreover, we can take $B'_K = 0$ if K is non-Archimedean with residue characteristic outside some finite set S' of prime numbers. By Claim 43, it follows that

$$\log \|\boldsymbol{G}(\boldsymbol{c})\|_{K^d \times K^{\frac{d(d-1)}{2}}} \geq A' \cdot M_{f_{\boldsymbol{c}}} + B'_K - A' \cdot \Delta_K$$

for all $\boldsymbol{c} \in K^{d-1}$. Now, for every $\boldsymbol{c} \in K^{d-1}$, we have

$$\left| \sigma_{d,j}^{(p)}([f_{\boldsymbol{c}}]) \right|_K \leq \left(\binom{N_d^{(p)}}{j} \right)_K \left(\max_{\lambda \in \Lambda_{f_{\boldsymbol{c}}}^{(p)}} |\lambda| \right)^j = \left(\binom{N_d^{(p)}}{j} \right)_K \exp\left(jp \cdot M_{f_{\boldsymbol{c}}}^{(p)} \right)$$

for all $p \geq 1$ and all $j \in \left\{1, \dots, N_d^{(p)}\right\}$ by the triangle inequality, and hence

$$\log \|\boldsymbol{G}(\boldsymbol{c})\|_{K^d \times K^{\frac{d(d-1)}{2}}} \leq \max_{\substack{p \in \{1,2\} \\ j \in \left\{1, \dots, N_d^{(p)}\right\}}} \left(\log \left(\binom{N_d^{(p)}}{j} \right)_K + jp \cdot M_{f_{\boldsymbol{c}}}^{(p)} \right) .$$

Therefore, setting

$$A = \frac{A'}{d(d-1)} \quad \text{and} \quad B_K = \min_{\substack{p \in \{1,2\} \\ j \in \left\{1, \dots, N_d^{(p)}\right\}}} \left(\frac{B'_K - A' \cdot \Delta_K - \log \left(\binom{N_d^{(p)}}{j} \right)_K}{jp} \right) ,$$

we have

$$\forall \boldsymbol{c} \in K^{d-1},\ \max \left\{ M_{f_{\boldsymbol{c}}}^{(1)}, M_{f_{\boldsymbol{c}}}^{(2)} \right\} \geq A \cdot M_{f_{\boldsymbol{c}}} + B_K .$$

Now, note that $A \in \mathbb{R}_{>0}$ does not depend on K and $B_K \in \mathbb{R}$ depends only on the restriction of $|.|_K$ to $\mathbb{Q}$. Furthermore, setting

$$S = S' \cup \left\{ q \text{ prime} : 2 \leq q \leq \max \left\{ d, \frac{d(d-1)}{2} \right\} \right\} ,$$

we have $B_K = 0$ if K is non-Archimedean with residue characteristic not in S.

To conclude, assume that $f \in \operatorname{Poly}_d(K)$ has leading coefficient $a_d \in K^*$. Choose any $(d-1)$th root $\alpha \in K^*$ of $d \cdot a_d$ and any fixed point $w \in K$ for f, and consider $\phi(z) = \alpha(z - w) \in \operatorname{Aff}(K)$. Then $\phi \centerdot f \in \operatorname{Poly}_d(K)$ has leading coefficient $\frac{1}{d}$ and it satisfies $\phi \centerdot f(0) = 0$, and hence $\phi \centerdot f = f_{\boldsymbol{c}}$, where $c_1, \dots, c_{d-1} \in K$ are the critical points for $\phi \centerdot f$ and $\boldsymbol{c} = (c_1, \dots, c_{d-1})$. Since $M_f = M_{f_{\boldsymbol{c}}}$ and $M_f^{(p)} = M_{f_{\boldsymbol{c}}}^{(p)}$ for each integer $p \geq 1$, we have $\max \left\{ M_f^{(1)}, M_f^{(2)} \right\} \geq A \cdot M_f + B_K$ by the discussion above. Thus, the proposition is proved. □

Remark 45. It follows immediately from Theorem B that we can take S to be the set of all primes less than or equal to d in the statement of Proposition 44.

Note that Corollary A.1 is simply a weaker version of Proposition 44. We shall now show that Corollary A.2 also follows directly from Proposition 44. To do this, let us first briefly recall various notions of height. We refer to [Lan83, Chapter 3], [Sil07, Chapter 3] and [Sil12, Chapter 5] for further details.

We denote here by $\mathbb{P}$ the set of all prime numbers. For $p \in \mathbb{P}$, denote by $\overline{\mathbb{Q}}_p$ the algebraic closure of the field $\mathbb{Q}_p$ of p-adic numbers and by $|.|_p$ the natural absolute value on $\overline{\mathbb{Q}}_p$. Also set $\overline{\mathbb{Q}}_\infty = \mathbb{C}$ and denote by $|.|_\infty$ the usual absolute value on $\overline{\mathbb{Q}}_\infty$. The *standard height* $h \colon \overline{\mathbb{Q}} \to \mathbb{R}_{\geq 0}$ is given by

$$h(t) = \frac{1}{[\mathbb{K} \colon \mathbb{Q}]} \sum_{p \in \mathbb{P} \cup \{\infty\}} \sum_{\sigma \colon \mathbb{K} \hookrightarrow \overline{\mathbb{Q}}_p} \log^+ |\sigma(t)|_p , \quad \text{with} \quad t \in \mathbb{K} \quad \text{and} \quad [\mathbb{K} \colon \mathbb{Q}] < +\infty .$$

Suppose that $f \in \operatorname{Poly}_d\left(\overline{\mathbb{Q}}\right)$. The *canonical height* $\widehat{h}_f \colon \overline{\mathbb{Q}} \to \mathbb{R}_{\geq 0}$ relative to f is defined by

$$\widehat{h}_f(z) = \lim_{n \to +\infty} \frac{1}{d^n} h\left(f^{\circ n}(z)\right) .$$

For every number field $\mathbb{K}$ such that $f \in \mathrm{Poly}_d(\mathbb{K})$ and every $z \in \mathbb{K}$, we have

$$\widehat{h}_f(z) = \frac{1}{[\mathbb{K} \colon \mathbb{Q}]} \sum_{p \in \mathbb{P} \cup \{\infty\}} \sum_{\sigma \colon \mathbb{K} \hookrightarrow \overline{\mathbb{Q}}_p} g_{\sigma(f)}\left(\sigma(z)\right) .$$

Now, denote by $\Gamma_f \subseteq \overline{\mathbb{Q}}$ the set of critical points for f and, for $\gamma \in \Gamma_f$, define ρ_γ to be its multiplicity as a critical point for f. The *critical height* H_f of f is given by

$$H_f = \sum_{\gamma \in \Gamma_f} \rho_\gamma \cdot \widehat{h}_f(\gamma) .$$

Proof of Corollary A.2. By Proposition 44, there exist $A' \in \mathbb{R}_{>0}$ and $B'_p \in \mathbb{R}$, for $p \in \mathbb{P} \cup \{\infty\}$, such that

$$\forall p \in \mathbb{P} \cup \{\infty\}, \forall g \in \mathrm{Poly}_d\left(\overline{\mathbb{Q}}_p\right), \max\left\{M_g^{(1)}, M_g^{(2)}\right\} \geq A' \cdot M_g + B'_p .$$

Moreover, we can take $B'_p = 0$ for all but finitely many $p \in \mathbb{P} \cup \{\infty\}$. Define

$$A = \frac{A'}{d-1} \in \mathbb{R}_{>0} \quad \text{and} \quad B = \sum_{p \in \mathbb{P} \cup \{\infty\}} B'_p \in \mathbb{R} .$$

Now, suppose that $f \in \mathrm{Poly}_d\left(\overline{\mathbb{Q}}\right)$. Fix a number field $\mathbb{K}$ containing the coefficients of f, its critical points and its multipliers at all its cycles with period 1 or 2. Then

$$\begin{aligned}
\max\left\{H_f^{(1)}, H_f^{(2)}\right\} &= \frac{1}{[\mathbb{K} \colon \mathbb{Q}]} \sum_{p \in \mathbb{P} \cup \{\infty\}} \sum_{\sigma \colon \mathbb{K} \hookrightarrow \overline{\mathbb{Q}}_p} \max_{\substack{q \in \{1,2\} \\ \lambda \in \Lambda_f^{(q)}}} \left(\frac{1}{q} \log^+ \left|\sigma(\lambda)\right|_p\right) \\
&\geq \frac{1}{[\mathbb{K} \colon \mathbb{Q}]} \sum_{p \in \mathbb{P} \cup \{\infty\}} \sum_{\sigma \colon \mathbb{K} \hookrightarrow \overline{\mathbb{Q}}_p} \max\left\{M_{\sigma(f)}^{(1)}, M_{\sigma(f)}^{(2)}\right\} \\
&\geq \frac{1}{[\mathbb{K} \colon \mathbb{Q}]} \sum_{p \in \mathbb{P} \cup \{\infty\}} \sum_{\sigma \colon \mathbb{K} \hookrightarrow \overline{\mathbb{Q}}_p} \left(A' \cdot M_{\sigma(f)} + B'_p\right) \\
&\geq \frac{A}{[\mathbb{K} \colon \mathbb{Q}]} \sum_{p \in \mathbb{P} \cup \{\infty\}} \sum_{\sigma \colon \mathbb{K} \hookrightarrow \overline{\mathbb{Q}}_p} \left(\sum_{\gamma \in \Gamma_f} \rho_\gamma \cdot g_{\sigma(f)}\left(\sigma(\gamma)\right)\right) + B \\
&= A \cdot H_f + B .
\end{aligned}$$

Thus, the corollary is proved. □

4. The non-Archimedean case

We shall adapt here the discussion of Section 3 in order to prove Theorem B in the non-Archimedean case.

Throughout this section, we fix an integer $d \geq 2$ and an algebraically closed field K of characteristic 0 equipped with a non-Archimedean absolute value $|.|$. We also assume that the residue characteristic of K either equals 0 or is greater than d, so that $|j| = 1$ for all $j \in \{1, \ldots, d\}$. In addition, we assume that $|.|$ is not the trivial absolute value, as Theorem B would be immediate otherwise. Note that we do not assume K to be complete here, although there is no gain in generality in not doing so since Theorem B clearly holds for K if it holds for its completion $\widehat{K}$.

4.1. **A few preliminaries on non-Archimedean analysis.** First, let us recall some basic facts about disks and polynomial maps in the non-Archimedean setting. We omit proofs and refer to [Ben19, Chapters 2 and 3] for further information.

In this section, we shall only work with finite unions of disks. Given $w \in K$ and $r \in \mathbb{R}_{>0}$, we denote by $D(w,r)$ and $\overline{D(w,r)}$ the open and closed disks of center w and radius r, respectively, which are given by

$$D(w,r) = \{z \in K : |z-w| < r\} \quad \text{and} \quad \overline{D(w,r)} = \{z \in K : |z-w| \leq r\} \,.$$

Note that a disk has a unique radius. In contrast, each point of a disk is a center. Although all disks are both open and closed topologically, we say here that a disk is *open* if it is of the form $D(w,r)$, with $w \in K$ and $r \in \mathbb{R}_{>0}$, and we say that it is *closed* if it is of the form $\overline{D(w,r)}$, with $w \in K$ and $r \in \mathbb{R}_{>0}$. Now, note that a disk is both open and closed if and only if its radius does not lie in $|K^*|$.

Suppose that U is a finite union of disks in K. Then U can be written uniquely as the union of finitely many pairwise disjoint disks $U_1, \dots, U_N$ in K, with $N \geq 0$. In addition, every disk contained in U is contained in U_j for some $j \in \{1, \dots, N\}$. These disks $U_1, \dots, U_N$ are called the *disk components* of U. The disk components of every finite union of open disks are all open. Similarly, the disk components of every finite union of closed disks are all closed.

Now, every nonconstant polynomial in $K[z]$ maps open disks to open disks and closed disks to closed disks. Given disks U, V in K, we say that a polynomial map $f\colon U \to V$ has *degree* $e \geq 1$ if every element of V has exactly e preimages under f in U, counting multiplicities.

Given a polynomial $f \in K[z]$ of degree $D \geq 1$ with leading coefficient $a_D \in K^*$ and $w \in K$, it is not hard to show that, for all $r \in \mathbb{R}_{>0}$ sufficiently large, we have

$$f\left(D(w,r)\right) = D\left(f(w), |a_D|\, r^D\right) \quad \text{and} \quad f\left(\overline{D(w,r)}\right) = \overline{D\left(f(w), |a_D|\, r^D\right)}$$

and the maps $f\colon D(w,r) \to D\left(f(w), |a_D|\, r^D\right)$ and $f\colon \overline{D(w,r)} \to \overline{D\left(f(w), |a_D|\, r^D\right)}$ have degree D. More generally, the result below describes precisely the images of disks under nonconstant polynomial maps in the non-Archimedean setting.

Lemma 46. *Suppose that $f \in K[z]$ has degree $D \geq 1$, $w \in K$ and $r \in \mathbb{R}_{>0}$. Set*

$$s = \max_{j \in \{1, \dots, D\}} \left| \frac{f^{(j)}(w)}{j!} \right| r^j \in \mathbb{R}_{>0} \,,$$

and define $e_{\min}$ and $e_{\max}$ to be the smallest and largest integers $j \in \{1, \dots, D\}$ such that $s = \left| \frac{f^{(j)}(w)}{j!} \right| r^j$, respectively. Then we have

$$f\left(D(w,r)\right) = D\left(f(w), s\right) \quad \text{and} \quad f\left(\overline{D(w,r)}\right) = \overline{D\left(f(w), s\right)} \,.$$

Moreover, the maps $f\colon D(w,r) \to D\left(f(w), s\right)$ and $f\colon \overline{D(w,r)} \to \overline{D\left(f(w), s\right)}$ have degrees $e_{\min}$ and $e_{\max}$, respectively.

We can also describe the preimages of disks under polynomial maps in the non-Archimedean setting.

Lemma 47. *Suppose that U, V are disks in K, $f\colon U \to V$ is a polynomial map of degree $e \geq 1$ and W is a disk contained in V. Then $f^{-1}(W)$ is a nonempty finite union of disks, and its disk components $U_1, \dots, U_N$, with $N \geq 1$, are all open if W*

is open and all closed if W is closed. Moreover, the map $f\colon U_j \to W$ has a degree $e_j \geq 1$ for each $j \in \{1, \dots, N\}$, and we have $e = \sum_{j=1}^{N} e_j$.

Remark 48. Note that, if $f \in K[z]$ is a polynomial of degree $e \geq 1$ and W is a disk in K, then the conclusion of Lemma 47 still holds. Indeed, in this case, there exist disks U, V in K such that $U = f^{-1}(V)$ and $W \subseteq V$, and $f\colon U \to V$ has degree e.

Finally, we have a non-Archimedean analogue of the Riemann–Hurwitz formula for disks, which relates the degree of a map to the number of its critical points.

Lemma 49. *Suppose that U, V are disks in K and $f\colon U \to V$ is a polynomial map of degree $e \geq 1$. Also assume that e is less than the residue characteristic of K if the latter is positive. Then $e = C + 1$, where $C \geq 0$ is the number of critical points for f in U, counting multiplicities.*

Remark 50. The assumption on the degree of the map in Lemma 49 is necessary, as the following shows: Suppose that $p \geq 2$ is a prime number. Then the algebraic closure $\overline{\mathbb{Q}}_p$ of the field $\mathbb{Q}_p$ of p-adic numbers is naturally a non-Archimedean field with residue characteristic p. The polynomial $f(z) = z^p - pz \in \overline{\mathbb{Q}}_p[z]$ maps $D(0,1)$ onto itself with degree p, while the critical points for f all lie outside $D(0,1)$. The polynomial $g(z) = z^{p+1} - z^p \in \overline{\mathbb{Q}}_p[z]$ maps $D(0,1)$ onto itself with degree p, while the critical points for g all lie in $D(0,1)$. In particular, the conclusion of Lemma 49 does not hold for the maps $f\colon D(0,1) \to D(0,1)$ and $g\colon D(0,1) \to D(0,1)$.

4.2. **The Green function of a polynomial map in the non-Archimedean setting.** Now, let us adapt our discussion of Green functions and maximal escape rates for complex polynomial maps to the non-Archimedean setting.

Suppose that $f \in \operatorname{Poly}_d(K)$. Recall that the *Green function* $g_f\colon K \to \mathbb{R}_{\geq 0}$ of f is given by

$$g_f(z) = \lim_{n \to +\infty} \frac{1}{d^n} \log^+ \left| f^{\circ n}(z) \right| .$$

This map g_f is well defined and satisfies $g_f \circ f = d \cdot g_f$. Moreover, for each $z \in K$, we have $g_f(z) = 0$ if and only if $\sup_{n \geq 0} \left| f^{\circ n}(z) \right| < +\infty$. Also recall that the *maximal escape rate* M_f of f is defined by

$$M_f = \max \left\{ g_f(c) : c \in K,\ f'(c) = 0 \right\} .$$

For every $\phi \in \operatorname{Aff}(K)$, we have $g_{\phi \bullet f} = g_f \circ \phi^{-1}$, and hence $M_{\phi \bullet f} = M_f$.

Thus, using conjugation, we may first restrict our attention to polynomials in a particular form. For $\boldsymbol{c} = (c_1, \dots, c_{d-1}) \in K^{d-1}$, define

$$f_{\boldsymbol{c}}(z) = \frac{1}{d} z^d + \sum_{j=1}^{d-1} \frac{(-1)^j \tau_j(\boldsymbol{c})}{d-j} z^{d-j} \in \operatorname{Poly}_d(K) ,$$

where $\tau_1(\boldsymbol{c}), \dots, \tau_{d-1}(\boldsymbol{c})$ denote the elementary symmetric functions of $c_1, \dots, c_{d-1}$, so that

$$f_{\boldsymbol{c}}(0) = 0 \quad \text{and} \quad f_{\boldsymbol{c}}'(z) = \prod_{j=1}^{d-1} (z - c_j) .$$

These polynomials have already been studied by Ingram in [Ing12]. Nevertheless, for completeness and to specify the values of certain constants in the present case, we include details.

For $\boldsymbol{c} = (c_1, \dots, c_{d-1}) \in K^{d-1}$, define

$$\|\boldsymbol{c}\| = \max_{j \in \{1,\dots,d-1\}} |c_j| \in \mathbb{R}_{\geq 0} \,.$$

Using the ultrametric triangle inequality and our assumption on the residue characteristic of K, we obtain the following:

Claim 51. We have $g_{f_{\boldsymbol{c}}}(z) \leq \log^+\left(\max\left\{\|\boldsymbol{c}\|, |z|\right\}\right)$ for all $\boldsymbol{c} \in K^{d-1}$ and all $z \in K$. Moreover, we have $g_{f_{\boldsymbol{c}}}(z) = \log^+|z|$ for all $\boldsymbol{c} \in K^{d-1}$ and all $z \in K \setminus \overline{D\left(0, \|\boldsymbol{c}\|\right)}$.

Proof. Note that the first assertion is simply a particular case of Claim 43. Thus, let us prove the second one. Suppose that $\boldsymbol{c} \in K^{d-1}$. For every $z \in K \setminus \overline{D\left(0, \|\boldsymbol{c}\|\right)}$, we have

$$\max_{j \in \{1,\dots,d-1\}} \left| \frac{(-1)^j \tau_j(\boldsymbol{c})}{d-j} z^{d-j} \right| \leq \max_{j \in \{1,\dots,d-1\}} \|\boldsymbol{c}\|^j |z|^{d-j} < |z|^d = \left| \frac{1}{d} z^d \right| ,$$

which yields $|f_{\boldsymbol{c}}(z)| = |z|^d$ by the ultrametric triangle inequality. By induction, we deduce that $\left|f_{\boldsymbol{c}}^{\circ n}(z)\right| = |z|^{d^n}$ for all $z \in K \setminus \overline{D\left(0, \max\left\{1, \|\boldsymbol{c}\|\right\}\right)}$ and all $n \geq 0$. As a result, for every $z \in K \setminus \overline{D\left(0, \max\left\{1, \|\boldsymbol{c}\|\right\}\right)}$, we have $\frac{1}{d^n} \log^+\left|f_{\boldsymbol{c}}^{\circ n}(z)\right| = \log^+|z|$ for all $n \geq 0$, which yields $g_{f_{\boldsymbol{c}}}(z) = \log^+|z|$ by letting $n \to +\infty$. Finally, we also have $g_{f_{\boldsymbol{c}}}(z) = \log^+|z|$ for each $z \in \overline{D(0,1)} \setminus \overline{D\left(0, \|\boldsymbol{c}\|\right)}$ by the first assertion of the claim. Thus, the claim is proved. □

Now, let us determine the maximal escape rates of these polynomials. To do so, we shall use Macaulay resultants to have an effective version of the Nullstellensatz for r homogeneous polynomials in r variables over a commutative ring, with $r \geq 1$. Thus, let us start by recalling a few necessary facts about resultants.

Suppose that R is a commutative ring and $P_1, \dots, P_r \in R\left[T_1, \dots, T_r\right]$ are homogeneous polynomials of degrees $e_1, \dots, e_r \geq 1$, respectively, with $r \geq 1$. Then there exists an element $\operatorname{res}\left(P_1, \dots, P_r\right) \in R$, called the *Macaulay resultant* of $P_1, \dots, P_r$, that satisfies the following:

- There are an integer $E \geq \max_{k \in \{1,\dots,r\}} e_k$ and some homogeneous polynomials $Q_{j,k} \in R\left[T_1, \dots, T_r\right]$ of degrees $E - e_k$, with $j, k \in \{1, \dots, r\}$, such that
$$\operatorname{res}\left(P_1, \dots, P_r\right) T_j^E = \sum_{k=1}^{r} P_k\left(T_1, \dots, T_r\right) Q_{j,k}\left(T_1, \dots, T_r\right)$$
for each $j \in \{1, \dots, r\}$.
- For any algebraically closed field Ω and any ring homomorphism $\varphi \colon R \to \Omega$, we have $\varphi\left(\operatorname{res}\left(P_1, \dots, P_r\right)\right) = 0$ if and only if the homogeneous polynomials $\varphi\left(P_1\right), \dots, \varphi\left(P_r\right) \in \Omega\left[T_1, \dots, T_r\right]$ have a common zero in $\Omega^r \setminus \{0\}$, where $\varphi \colon R\left[T_1, \dots, T_r\right] \to \Omega\left[T_1, \dots, T_r\right]$ denotes the unique ring homomorphism that extends $\varphi \colon R \to \Omega$ and satisfies $\varphi\left(T_j\right) = T_j$ for all $j \in \{1, \dots, r\}$.

We refer to [Lan02, Chapter IX, Section 3] for further details about resultants.

We now return to the study of the polynomials $f_{\boldsymbol{c}} \in \operatorname{Poly}_d(K)$, with $\boldsymbol{c} \in K^{d-1}$. Consider the subring $A = \mathbb{Z}\left[\frac{1}{2}, \dots, \frac{1}{d}\right]$ of K. For each $a \in A$, we have $|a| \leq 1$, with equality holding if and only if a is not divisible by the residue characteristic of K. For $j \in \{1, \dots, d-1\}$, also consider the polynomial $F_j \in A\left[T_1, \dots, T_{d-1}\right]$ given by $F_j(\boldsymbol{c}) = f_{\boldsymbol{c}}\left(c_j\right)$ for all $\boldsymbol{c} = (c_1, \dots, c_{d-1}) \in K^{d-1}$. For every $j \in \{1, \dots, d-1\}$, the polynomial F_j is homogeneous of degree d. Furthermore, we have the following:

Claim 52. We have $|\mathrm{res}\,(F_1, \dots, F_{d-1})| = 1$.

Proof. Suppose that $p > d$ is a prime number. Let us show that $\mathrm{res}\,(F_1, \dots, F_{d-1})$ is not divisible by p in A. Note that A/pA is the field $\mathbb{F}_p$ with p elements. Thus, denoting by $\overline{\mathbb{F}}_p$ the algebraic closure of $\mathbb{F}_p$, we have a natural ring homomorphism $\varphi\colon A \to \overline{\mathbb{F}}_p$. Then $\mathrm{res}\,(F_1, \dots, F_{d-1}) \in pA$ if and only if $\varphi\,(\mathrm{res}\,(F_1, \dots, F_{d-1})) = 0$, which occurs if and only if $\varphi\,(F_1)\,, \dots, \varphi\,(F_{d-1})$ have a common zero in $\overline{\mathbb{F}}_p^{d-1} \setminus \{0\}$. Suppose that $\mathfrak{c} = (c_1, \dots, c_{d-1}) \in \overline{\mathbb{F}}_p^{d-1}$ is a common zero of $\varphi\,(F_1)\,, \dots, \varphi\,(F_{d-1})$. We shall prove that $\mathfrak{c} = 0$. Define

$$\mathfrak{f}(z) = \frac{1}{d} z^d + \sum_{j=1}^{d-1} \frac{(-1)^j \tau_j(\mathfrak{c})}{d-j} z^{d-j} \in \overline{\mathbb{F}}_p[z]\,,$$

where $\tau_1(\mathfrak{c}), \dots, \tau_{d-1}(\mathfrak{c})$ denote the elementary symmetric functions of $c_1, \dots, c_{d-1}$. We have $\mathfrak{f}'(z) = \prod_{j=1}^{d-1} (z - c_j)$ and $\mathfrak{f}\,(c_j) = \varphi\,(F_j)\,(\mathfrak{c}) = 0$ for each $j \in \{1, \dots, d-1\}$. Now, define $\Gamma = \{c_1, \dots, c_{d-1}\}$ and, for $\gamma \in \Gamma$, denote by ρ_γ the number of indices $j \in \{1, \dots, d-1\}$ such that $c_j = \gamma$. We have $\mathfrak{f}'(z) = \prod_{\gamma \in \Gamma} (z - \gamma)^{\rho_\gamma}$ and $\mathfrak{f}(\gamma) = 0$ for each $\gamma \in \Gamma$. As a result, as $\overline{\mathbb{F}}_p$ has characteristic $p > d$, each $\gamma \in \Gamma$ has multiplicity $\rho_\gamma + 1$ as a preimage of 0 under $\mathfrak{f}$ and every other preimage of 0 has multiplicity 1. It follows that

$$d = r + \sum_{\gamma \in \Gamma} (\rho_\gamma + 1) = r + d - 1 + s\,,$$

where $r \geq 0$ is the number of preimages of 0 under $\mathfrak{f}$ that are not in Γ and $s \geq 1$ is the cardinality of Γ, which yields $r = 0$ and $s = 1$. Therefore, as $\mathfrak{f}(0) = 0$, we have $\Gamma = \{0\}$, and hence $\mathfrak{c} = 0$. Thus, we have proved that $\mathrm{res}\,(F_1, \dots, F_{d-1}) \in A \setminus pA$ for each prime number $p > d$. In particular, $\mathrm{res}\,(F_1, \dots, F_{d-1}) \in A$ is not divisible by the residue characteristic of K. This completes the proof of the claim. □

This allows us to determine the maximal escape rate $M_{f_{\boldsymbol{c}}}$ of $f_{\boldsymbol{c}}$, with $\boldsymbol{c} \in K^{d-1}$.

Claim 53. We have $M_{f_{\boldsymbol{c}}} = \log^+\|\boldsymbol{c}\|$ for all $\boldsymbol{c} \in K^{d-1}$.

Proof. Suppose that $\boldsymbol{c} = (c_1, \dots, c_{d-1}) \in K^{d-1}$. We have $g_{f_{\boldsymbol{c}}}\,(c_j) \leq \log^+\|\boldsymbol{c}\|$ for all $j \in \{1, \dots, d-1\}$ by the first assertion of Claim 51, and hence $M_{f_{\boldsymbol{c}}} \leq \log^+\|\boldsymbol{c}\|$. It remains to show that $M_{f_{\boldsymbol{c}}} \geq \log^+\|\boldsymbol{c}\|$. If $\|\boldsymbol{c}\| \leq 1$, this is immediate. Thus, assume now that $\|\boldsymbol{c}\| > 1$. There exist some integer $D \geq d$ and homogeneous polynomials $G_{j,k} \in A\,[T_1, \dots, T_{d-1}]$ of degree $D - d$, with $j, k \in \{1, \dots, d-1\}$, such that

$$\mathrm{res}\,(F_1, \dots, F_{d-1})\, c_j^D = \sum_{k=1}^{d-1} f_{\boldsymbol{c}}\,(c_k)\, G_{j,k}(\boldsymbol{c})$$

for each $j \in \{1, \dots, d-1\}$. Therefore, by Claim 52, we have

$$|c_j|^D \leq \max_{k \in \{1, \dots, d-1\}} |f_{\boldsymbol{c}}\,(c_k)\, G_{j,k}(\boldsymbol{c})| \leq \left(\max_{k \in \{1, \dots, d-1\}} |f_{\boldsymbol{c}}\,(c_k)| \right) \|\boldsymbol{c}\|^{D-d}$$

for all $j \in \{1, \dots, d-1\}$, and hence $\max_{k \in \{1, \dots, d-1\}} |f_{\boldsymbol{c}}\,(c_k)| \geq \|\boldsymbol{c}\|^d$. Thus, there exists $k \in \{1, \dots, d-1\}$ such that $|f_{\boldsymbol{c}}\,(c_k)| \geq \|\boldsymbol{c}\|^d$. By the second assertion of Claim 51, as $\|\boldsymbol{c}\| > 1$, it follows that

$$d \cdot g_{f_{\boldsymbol{c}}}(c_j) = g_{f_{\boldsymbol{c}}}(f_{\boldsymbol{c}}(c_j)) = \log^+ |f_{\boldsymbol{c}}(c_j)| \geq d \cdot \log^+ \|\boldsymbol{c}\| .$$

Thus, we have $M_{f_{\boldsymbol{c}}} \geq \log^+ \|\boldsymbol{c}\|$, and the claim is proved. □

From the discussion above, we now derive results about the Green functions of arbitrary polynomials in $\mathrm{Poly}_d(K)$.

Lemma 54. *Suppose that $f \in \mathrm{Poly}_d(K)$ has leading coefficient $a_d \in K^*$. Then, for each $\eta \in \mathbb{R}_{>0}$, the set $\{g_f < \eta\}$ is a nonempty finite union of open disks and the set $\{g_f \leq \eta\}$ is a nonempty finite union of closed disks. Moreover, $\{g_f < \eta\}$ is an open disk of radius $|a_d|^{\frac{-1}{d-1}} \exp(\eta)$ for all $\eta \in (M_f, +\infty)$. In addition, $\{g_f \leq \eta\}$ is a closed disk of radius $|a_d|^{\frac{-1}{d-1}} \exp(\eta)$ for all $\eta \in [M_f, +\infty)$.*

Proof. Choose a $(d-1)$th root $\alpha \in K^*$ of $d \cdot a_d$ and a fixed point $w \in K$ for f, and define $\phi(z) = \alpha(z - w) \in \mathrm{Aff}(K)$. Then $\phi \centerdot f \in \mathrm{Poly}_d(K)$ has leading coefficient $\frac{1}{d}$ and satisfies $\phi \centerdot f(0) = 0$, and therefore $\phi \centerdot f = f_{\boldsymbol{c}}$, where $c_1, \dots, c_{d-1} \in K$ are the critical points for $\phi \centerdot f$ and $\boldsymbol{c} = (c_1, \dots, c_{d-1})$. Now, by Claims 51 and 53, we have $\{g_{f_{\boldsymbol{c}}} < \eta\} = D(0, \exp(\eta))$ for all $\eta \in (M_{f_{\boldsymbol{c}}}, +\infty)$. Moreover, we have $g_{f_{\boldsymbol{c}}} = g_f \circ \phi^{-1}$ and $M_{f_{\boldsymbol{c}}} = M_f$ by conjugation. Therefore, as $|\alpha| = |a_d|^{\frac{1}{d-1}}$, we have

$$\forall \eta \in (M_f, +\infty), \{g_f < \eta\} = \phi^{-1}(\{g_{f_{\boldsymbol{c}}} < \eta\}) = D\left(w, |a_d|^{\frac{-1}{d-1}} \exp(\eta)\right) .$$

Similarly, we have $\{g_{f_{\boldsymbol{c}}} \leq \eta\} = \overline{D(0, \exp(\eta))}$ for each $\eta \in [M_{f_{\boldsymbol{c}}}, +\infty)$ by Claims 51 and 53, and hence

$$\forall \eta \in [M_f, +\infty), \{g_f \leq \eta\} = \phi^{-1}(\{g_{f_{\boldsymbol{c}}} \leq \eta\}) = \overline{D\left(w, |a_d|^{\frac{-1}{d-1}} \exp(\eta)\right)} .$$

Finally, suppose that $\eta \in \mathbb{R}_{>0}$. There exists an integer $k \geq 0$ such that $d^k \eta > M_f$. Then $\left\{g_f < d^k \eta\right\}$ is an open disk in K by the previous discussion. By Lemma 47, it follows that

$$\{g_f < \eta\} = \left(f^{\circ k}\right)^{-1}\left(\left\{g_f < d^k \eta\right\}\right)$$

is a nonempty finite union of open disks. Similarly, $\left\{g_f \leq d^k \eta\right\}$ is a closed disk in K by the previous discussion, and hence

$$\{g_f \leq \eta\} = \left(f^{\circ k}\right)^{-1}\left(\left\{g_f \leq d^k \eta\right\}\right)$$

is a nonempty finite union of closed disks by Lemma 47. This completes the proof of the lemma. □

Finally, we also have the following non-Archimedean analogue of Lemma 19:

Lemma 55. *Suppose that $f \in \mathrm{Poly}_d(K)$ satisfies $M_f > 0$, and denote by $a_d \in K^*$ its leading coefficient and by $C \geq 1$ the number of critical points $c \in K$ for f such that $g_f(c) = M_f$, counting multiplicities. Then $\{g_f < M_f\}$ has exactly $C + 1$ disk components, and these are all open disks of radius $|a_d|^{\frac{-1}{d-1}} \exp(M_f)$.*

Proof. By Lemma 54, the set $\{g_f < M_f\}$ is a nonempty finite union of open disks. Denote here by $U_1, \dots, U_N$, with $N \geq 1$, its disk components. For $j \in \{1, \dots, N\}$, denote by $C_j \geq 0$ the number of critical points for f in U_j, counting multiplicities. Note that $\sum_{j=1}^{N} C_j = d - 1 - C$. Now, as $\{g_f < d \cdot M_f\}$ is a disk by Lemma 54 and

$$\{g_f < M_f\} = f^{-1}\left(\{g_f < d \cdot M_f\}\right) ,$$

the map $f \colon U_j \to \{g_f < d \cdot M_f\}$ has a degree $d_j \geq 1$ for all $j \in \{1, \ldots, N\}$, and we have $d = \sum_{j=1}^{N} d_j$, by Lemma 47. In addition, $d_j = C_j + 1$ for each $j \in \{1, \ldots, N\}$ by Lemma 49. Therefore, we have

$$d = \sum_{j=1}^{N} (C_j + 1) = d - 1 - C + N ,$$

and hence $N = C + 1$, as desired.

Finally, note that $\{g_f \leq M_f\}$ is a disk of radius $|a_d|^{\frac{-1}{d-1}} \exp(M_f)$ by Lemma 54. Moreover, for every $j \in \{1, \ldots, C+1\}$, we have

$$f(U_j) = \{g_f < d \cdot M_f\} \quad \text{and} \quad f(\{g_f \leq M_f\}) = \{g_f \leq d \cdot M_f\} ,$$

and these are two disks of the same radius by Lemma 54. By Lemma 46, it follows that U_j also has radius $|a_d|^{\frac{-1}{d-1}} \exp(M_f)$ for all $j \in \{1, \ldots, C+1\}$. This completes the proof of the lemma. ☐

4.3. **A combinatorial argument.** Now, let us obtain an analogue of Lemma 25, which plays a key role in our proof of Theorem B in the Archimedean case.

To do this, we shall first prove the two-islands lemma below, which is the non-Archimedean counterpart of Lemma 23.

Lemma 56. *Suppose that U, V are disks, $f \colon U \to V$ is a polynomial map of degree $e \geq 1$ and V_1, V_2 are disjoint disks contained in V. Also assume that e is less than the residue characteristic of K if the latter is positive. Then there exist $j \in \{1, 2\}$ and a disk component U_j of $f^{-1}(V_j)$ such that f induces a bijection from U_j to V_j.*

Proof. For $j \in \{1, 2\}$, denote here by $C_j \geq 0$ the number of critical points for f in $f^{-1}(V_j)$, counting multiplicities. Then, by Lemma 49, we have

$$e = C + 1 \geq C_1 + C_2 + 1 \geq 2 \min\{C_1, C_2\} + 1 ,$$

where $C \geq 0$ is the number of critical points for f in U, counting multiplicities. By Lemma 47, the set $f^{-1}(V_j)$ is a nonempty finite union of disks for each $j \in \{1, 2\}$. For $j \in \{1, 2\}$, denote by $U_j^{(1)}, \ldots, U_j^{(N_j)}$ its disk components, with $N_j \geq 1$. Then, by Lemma 47, for each $j \in \{1, 2\}$, the map $f \colon U_j^{(\ell)} \to V_j$ has a degree $e_j^{(\ell)} \geq 1$ for all $\ell \in \{1, \ldots, N_j\}$, and we have $e = \sum_{\ell=1}^{N_j} e_j^{(\ell)}$. In addition, for all $j \in \{1, 2\}$ and all $\ell \in \{1, \ldots, N_j\}$, we have $e_j^{(\ell)} = C_j^{(\ell)} + 1$ by Lemma 49, where $C_j^{(\ell)} \geq 0$ denotes the number of critical points for f in $U_j^{(\ell)}$, counting multiplicities. Thus, we have

$$\forall j \in \{1, 2\},\ e = \sum_{\ell=1}^{N_j} \left(C_j^{(\ell)} + 1\right) = C_j + N_j .$$

Therefore, as $e \geq 2 \min \{C_1, C_2\} + 1$, there exists $j \in \{1, 2\}$ such that $C_j < N_j$. As a result, $C_j^{(\ell)} = 0$ for some $\ell \in \{1, \ldots, N_j\}$, and we have $e_j^{(\ell)} = 1$. Thus, f induces a bijection from $U_j^{(\ell)}$ to V_j, and the lemma is proved. □

Remark 57. The assumptions that $f\colon U \to V$ is surjective and that its degree e is finite and satisfies a certain condition related to the residue characteristic of K are essential in our proof of Lemma 56. In [Ben03], Benedetto proved a more involved two-islands theorem for holomorphic functions on a disk in an algebraically closed field that is complete with respect to a non-Archimedean and nontrivial absolute value. In [Ben08], Benedetto also proved a non-Archimedean four-islands theorem for meromorphic functions. We refer the reader to these articles for more details.

Finally, we have the result below, which is completely analogous to Lemma 25.

Lemma 58. *Suppose that $f \in \mathrm{Poly}_d(K)$ satisfies $M_f > 0$. Then one of the following two conditions is satisfied:*

(1) *there exists a disk component U of $\left\{g_f < \frac{M_f}{d}\right\}$ such that f induces a bijection from U onto the disk component V of $\{g_f < M_f\}$ containing U;*
(2) *for all distinct disk components V, V' of $\{g_f < M_f\}$, there exists a disk component U of $\left\{g_f < \frac{M_f}{d}\right\}$ contained in V such that f induces a bijection from U onto V'.*

In addition, if $d \in \{2, 3\}$, then there exists a disk component V of $\{g_f < M_f\}$ such that f induces a bijection from V onto $\{g_f < d \cdot M_f\}$.

Proof. Assume here that the condition (1) does not hold, and let us show that the condition (2) is satisfied. By Lemma 54, $\left\{g_f < \frac{M_f}{d}\right\}$ and $\{g_f < M_f\}$ are nonempty finite unions of disks. Now, assume that V, V' are two distinct disk components of $\{g_f < M_f\}$. It follows from Lemma 47 that $f^{-1}(V)$ and $f^{-1}(V')$ are both unions of disk components of $\left\{g_f < \frac{M_f}{d}\right\}$. Now, as $\{g_f < d \cdot M_f\}$ is a disk by Lemma 54, the induced map $f\colon V \to \{g_f < d \cdot M_f\}$ has a degree $e \in \{1, \ldots, d\}$ by Lemma 47. Moreover, no disk component of $\left\{g_f < \frac{M_f}{d}\right\}$ contained in V is mapped bijectively onto V by f by hypothesis. Therefore, by Lemma 56, f maps a disk component of $\left\{g_f < \frac{M_f}{d}\right\}$ contained in V bijectively onto V'. Thus, the desired result is proved.

Finally, assume that $d \in \{2, 3\}$. By Lemma 54, $\{g_f < M_f\}$ is a nonempty finite union of disks and $\{g_f < d \cdot M_f\}$ is a disk. In fact, by Lemma 55, $\{g_f < M_f\}$ has several disk components $V_1, \ldots, V_N$, with $N \geq 2$. By Lemma 47, the induced map $f\colon V_j \to \{g_f < d \cdot M_f\}$ has a degree $d_j \geq 1$ for each $j \in \{1, \ldots, N\}$, and $d = \sum_{j=1}^{N} d_j$. Therefore, as $d \leq 3$, there exists $j \in \{1, \ldots, N\}$ such that $d_j = 1$, and f induces a bijection from V_j onto $\{g_f < d \cdot M_f\}$. This completes the proof of the lemma. □

4.4. **Multipliers and maximal escape rates.** Here, let us relate multipliers at periodic points to maximal escape rates under certain combinatorial assumptions. More precisely, let us obtain a non-Archimedean analogue of Lemma 32.

In the non-Archimedean setting, ratios of radii of disks play the role of moduli of complex annuli. Thus, we have the well-known result below, which follows from Lemma 46 and is the non-Archimedean counterpart of Lemma 31.

Lemma 59. *Suppose that $U \subsetneq V$ are disks and $f\colon U \to V$ is a bijective polynomial map. Then f has a unique fixed point $z_0 \in U$ and $\left|f'\left(z_0\right)\right| = \frac{s}{r}$, where $r, s \in \mathbb{R}_{>0}$ are the radii of U, V, respectively.*

Proof. Note that $s > r$ since $U \subsetneq V$ by hypothesis and U and V are both open or both closed by Lemma 46. Now, choose $w \in f^{-1}(U)$. By Lemma 46, as $w \in U$ and $f\colon U \to V$ is bijective, we have $|f'(w)| = \frac{s}{r}$ and $s \geq \left|\frac{f^{(j)}(w)}{j!}\right| r^j$ for each $j \geq 2$, with strict inequality if U is closed. Now, define $g(z) = f(z) - z \in K[z]$. Then we have $|g'(w)| = |f'(w) - 1| = \frac{s}{r}$ since $|f'(w)| = \frac{s}{r} > 1$. Moreover, for each $j \geq 2$, we have $s \geq \left|\frac{g^{(j)}(w)}{j!}\right| r^j$, with strict inequality if U is closed, since $g^{(j)}(w) = f^{(j)}(w)$. As a result, $g(U)$ is a disk of radius s and the induced map $g\colon U \to g(U)$ is bijective by Lemma 46. Moreover, we have $|g(w)| = |f(w) - w| \leq r$ as $w \in f^{-1}(U)$, and hence $0 \in g(U)$. Therefore, the map $f\colon U \to V$ has a unique fixed point $z_0 \in U$. Finally, as $f\colon U \to V$ is bijective, we have $\left|f'\left(z_0\right)\right| = \frac{s}{r}$ by Lemma 46. This completes the proof of the lemma. □

We also have the well-known result below, which follows easily from Lemma 46 and is a non-Archimedean analogue of Grötzsch's inequality.

Lemma 60. *Suppose that $f \in K[z]$ is not constant and $U \subseteq V$ are disks. Then*

$$\left(\frac{R}{r}\right)^e \leq \frac{S}{s} \leq \left(\frac{R}{r}\right)^E ,$$

where r, R, s, S are the radii of $U, V, f(U), f(V)$, respectively, and e and E are the degrees of $f\colon U \to f(U)$ and $f\colon V \to f(V)$, respectively.

Proof. Choose $w \in U$. Then U, V are disks of center w and radii r, R, respectively. Therefore, by Lemma 46, we have

$$s = \left|\frac{f^{(e)}(w)}{e!}\right| r^e \geq \left|\frac{f^{(E)}(w)}{E!}\right| r^E \quad \text{and} \quad S = \left|\frac{f^{(E)}(w)}{E!}\right| R^E \geq \left|\frac{f^{(e)}(w)}{e!}\right| R^e .$$

This completes the proof of the lemma. □

Finally, we obtain the result below, which is the non-Archimedean counterpart of Lemma 32. Note that, in the present context, we also have an upper bound on the absolute values of multipliers.

Lemma 61. *Suppose that $f \in \mathrm{Poly}_d(K)$ satisfies $M_f > 0$, $\eta \geq M_f$, $U_0, \dots, U_{p-1}$ are disk components of $\left\{g_f < \frac{\eta}{d^k}\right\}$, with $k \geq 0$ and $p \geq 1$, $V_0, \dots, V_{p-1}$ are the disk components of $\left\{g_f < \frac{\eta}{d^{k-1}}\right\}$ containing $U_0, \dots, U_{p-1}$, respectively, and f induces a bijection from U_j to $V_{j+1 \pmod{p}}$ for all $j \in \{0, \dots, p-1\}$. Then $f^{\circ p}$ has a unique fixed point $z_0 \in K$ such that $f^{\circ j}\left(z_0\right) \in U_j$ for all $j \in \{0, \dots, p-1\}$. Furthermore, we have*

$$(d-1)\left(\sum_{j=0}^{p-1} \frac{1}{e_j}\right)\eta \leq \log\left|\left(f^{\circ p}\right)'\left(z_0\right)\right| \leq (d-1)\left(\sum_{j=0}^{p-1} \frac{1}{d_j}\right)\eta ,$$

where d_j and e_j denote the degrees of the induced maps $f^{\circ k}\colon \overline{U}_j \to \{g_f \leq \eta\}$ and $f^{\circ k}\colon V_j \to \{g_f < d \cdot \eta\}$, respectively, and $\overline{U}_j$ is the disk component of $\left\{g_f \leq \frac{\eta}{d^k}\right\}$ containing U_j for all $j \in \{0, \dots, p-1\}$.

Proof. For $j \in \{0, \dots, p-1\}$, define $f_j \colon U_j \to V_{j+1 \pmod p}$ to be the bijective map induced by f. For $j \in \{0, \dots, p\}$, define

$$W_j = (f_{j-1} \circ \dots \circ f_0)^{-1} \left(V_{j \pmod p}\right) = (f_{j-2} \circ \dots \circ f_0)^{-1} (U_{j-1}) ,$$

where $W_0 = V_0$ and $W_1 = U_0$ by convention. It follows from Lemma 47 that W_j is a disk for all $j \in \{0, \dots, p\}$. In addition, for each $j \in \{0, \dots, p-1\}$, we have

$$f^{\circ k}(U_j) \subseteq \{g_f \leq \eta\} \subsetneq \{g_f < d \cdot \eta\} = f^{\circ k}(V_j)$$

by Lemmas 47 and 54, which yields $U_j \subsetneq V_j$, and hence $W_{j+1} \subsetneq W_j$. As a result, by Lemma 59, the map $f_{p-1} \circ \dots \circ f_0 \colon W_p \to W_0$ has a unique fixed point $z_0 \in W_p$ and we have

$$\left|(f^{\circ p})'(z_0)\right| = \frac{\rho_0}{\rho_p} = \prod_{j=0}^{p-1} \frac{\rho_j}{\rho_{j+1}} ,$$

where $\rho_0, \dots, \rho_p$ denote the radii of $W_0, \dots, W_p$, respectively. Now, note that z_0 is the unique fixed point for $f^{\circ p}$ that satisfies $f^{\circ j}(z_0) \in U_j$ for all $j \in \{0, \dots, p-1\}$. Thus, it remains to prove the desired inequalities. For $j \in \{0, \dots, p-1\}$, define r_j and R_j to be the radii of U_j and V_j, respectively. Then, for all $j \in \{0, \dots, p-1\}$, we have $\frac{\rho_j}{\rho_{j+1}} = \frac{R_j}{r_j}$ by Lemma 60 since $f_{j-1} \circ \dots \circ f_0$ maps bijectively W_j onto V_j and W_{j+1} onto U_j. Thus, it suffices to prove that $\left(\frac{d-1}{e_j}\right)\eta \leq \log\left(\frac{R_j}{r_j}\right) \leq \left(\frac{d-1}{d_j}\right)\eta$ for all $j \in \{0, \dots, p-1\}$. Suppose that $j \in \{0, \dots, p-1\}$. As $d \cdot \eta > M_f$, we have $f^{\circ(k+1)}(U_j) = \{g_f < d \cdot \eta\}$ and $f^{\circ(k+1)}\left(\overline{U}_j\right) = \{g_f \leq d \cdot \eta\}$ by Lemmas 47 and 54, and these are disks of the same radius by Lemma 54. As a result, the disk $\overline{U}_j$ also has radius r_j by Lemma 46. Therefore, as $f^{\circ k}$ maps $\overline{U}_j$ onto $\{g_f \leq \eta\}$ with degree d_j and V_j onto $\{g_f < d \cdot \eta\}$ with degree e_j, we have

$$\left(\frac{R_j}{r_j}\right)^{d_j} \leq \exp\left((d-1)\eta\right) \leq \left(\frac{R_j}{r_j}\right)^{e_j}$$

by Lemmas 54 and 60. This completes the proof of the lemma. □

Remark 62. To prove Theorem B in the non-Archimedean case, we shall only use Lemma 61 with $\eta = M_f$, $k \in \{0, 1\}$ and $p \in \{1, 2\}$ to only derive lower bounds on the absolute values of multipliers at certain small cycles. Nonetheless, our general statement of Lemma 61 also allows us to show that the bounds in Theorem B are optimal (see Propositions 65 and 66) and to obtain a lower bound on the absolute values of multipliers at all cycles in terms of the periods and the minimum of the Green function on the set of critical points (see Proposition 83).

4.5. **Proof of Theorem B in the non-Archimedean case.** Now, let us derive Theorem B in the non-Archimedean case from Lemmas 58 and 61. This is similar to our proof of Theorem B in the Archimedean case.

Proof of Theorem B in the non-Archimedean case. Assume here that K is an algebraically closed field of characteristic 0 that is equipped with a non-Archimedean absolute value $|.|$ with residue characteristic 0 or greater than d and $f \in \mathrm{Poly}_d(K)$. Note that the desired result is immediate if the absolute value $|.|$ is trivial. Thus, from now on, assume that $|.|$ is not trivial.

First, suppose that $M_f = 0$. Denote by $\lambda_1, \dots, \lambda_d$ the multipliers of f at all its fixed points repeated according to their multiplicities. Define $\sigma_1, \dots, \sigma_d$ to be the elementary symmetric functions of $\lambda_1, \dots, \lambda_d$. We have $d + \sum_{j=1}^{d} (-1)^j (d-j)\sigma_j = 0$ by the holomorphic fixed-point formula. Therefore, since $|d| = 1$, we have $|\sigma_j| \geq 1$ for some $j \in \{1, \dots, d\}$ by the ultrametric triangle inequality. As a result, we have $|\lambda_k| \geq 1$ for some $k \in \{1, \dots, d\}$ by the ultrametric triangle inequality. This shows that $M_f^{(1)} \geq 0$, as desired.

Thus, from now on, assume that $M_f > 0$. By Lemma 54, $\{g_f < M_f\}$ is a finite union of disks and $\{g_f < d \cdot M_f\}$ is a disk. Moreover, $\{g_f < M_f\}$ has several disk components $V_1, \dots, V_N$, with $N \geq 2$, by Lemma 55. Now, by Lemma 47, the map $f \colon V_j \to \{g_f < d \cdot M_f\}$ has some degree $d_j \geq 1$ for all $j \in \{1, \dots, N\}$, and we have $d = \sum_{j=1}^{N} d_j$. To conclude, let us consider three cases.

Suppose that $d_j = 1$ for some $j \in \{1, \dots, N\}$. Note that this holds if $d \in \{2, 3\}$. Then f induces a bijection from V_j onto $\{g_f < d \cdot M_f\}$. Therefore, by Lemma 61, f has a unique fixed point $z_0 \in V_j$ and we have $\log |f'(z_0)| \geq (d-1) M_f$. Thus, we have $M_f^{(1)} \geq (d-1) M_f$.

Now, suppose that the condition (1) of Lemma 58 is satisfied and $d_j \geq 2$ for all $j \in \{1, \dots, N\}$. Then there exist some $j \in \{1, \dots, N\}$ and a disk component U_j of $\left\{g_f < \frac{M_f}{d}\right\}$ contained in V_j such that f maps bijectively U_j to V_j. As a result, by Lemma 61, f has a unique fixed point $z_0 \in U_j$ and we have $\log |f'(z_0)| \geq \frac{d-1}{d_j} M_f$. Moreover, $d_j = d - \sum_{k \neq j} d_k \leq d - 2$. Thus, we have $M_f^{(1)} \geq \frac{d-1}{d-2} M_f$.

Finally, suppose that the condition (2) of Lemma 58 is satisfied. Then there are disk components U_1, U_2 of $\left\{g_f < \frac{M_f}{d}\right\}$ contained in V_1, V_2, respectively, such that f maps bijectively U_1 to V_2 and U_2 to V_1. As a result, by Lemma 61, the map $f^{\circ 2}$ has a unique fixed point $z_0 \in K$ such that $z_0 \in U_1$ and $f(z_0) \in U_2$ and we have

$$\log \left| \left(f^{\circ 2}\right)'(z_0) \right| \geq (d-1) \left(\frac{1}{d_1} + \frac{1}{d_2} \right) M_f \geq (d-1) \left(\frac{1}{d_1} + \frac{1}{d - d_1} \right) M_f \,.$$

Moreover, $\frac{d-1}{2} \left(\frac{1}{d_1} + \frac{1}{d-d_1} \right) \geq C_d$. Thus, we have $M_f^{(2)} \geq C_d \cdot M_f$. This completes the proof of the theorem. □

Remark 63. As in the complex setting, one can strengthen Lemmas 56 and 58 and deduce that $M_f^{(2)} \geq 2 M_f$ for all $f \in \mathrm{Poly}_d(K)$ such that $M_f > 0$.

4.6. **Sharpness of the bounds.** To end this section, let us show that the bounds in Theorem B are sharp in some sense. We shall first study the non-Archimedean case and then deduce that our bounds are also optimal in the complex setting. To avoid making our discussion too long, we omit the simpler cases where $d \in \{2, 3\}$.

We shall use the following result from non-Archimedean dynamics:

Lemma 64. *Suppose that $U \subseteq V$ are disks and $f \colon U \to V$ is a polynomial map of degree $e \geq 2$ whose critical points all lie in $\bigcap_{n \geq 0} \left(f^{\circ n}\right)^{-1}(V)$. Also assume that e is less than the residue characteristic of K if the latter is positive. Then we have $\left|\left(f^{\circ p}\right)'(z_0)\right| \leq 1$ for each periodic point $z_0 \in K$ for f with period $p \geq 1$.*

Proof. First, note that f has exactly $e - 1$ critical points in U, counting multiplicities, by Lemma 49. Now, assume that $\left(f^{\circ n}\right)^{-1}(V)$ is a disk for some $n \geq 0$. Then, by Lemma 47, $\left(f^{\circ(n+1)}\right)^{-1}(V)$ is the union of finitely many pairwise disjoint disks $W_1, \dots, W_N$, with $N \geq 1$, the induced map $f \colon W_j \to \left(f^{\circ n}\right)^{-1}(V)$ has some degree $e_j \geq 1$ for each $j \in \{1, \dots, N\}$, and we have $e = \sum_{j=1}^{N} e_j$. For $j \in \{1, \dots, N\}$, denote by $C_j \geq 0$ the number of critical points for f in W_j, counting multiplicities. Then $\sum_{j=1}^{N} C_j = e - 1$ by assumption. In addition, $e_j = C_j + 1$ for each $j \in \{1, \dots, N\}$ by Lemma 49. As a result, we have $e = \sum_{j=1}^{N} (C_j + 1) = e - 1 + N$, which yields $N = 1$ and $e_1 = e$. Thus, $\left(\left(f^{\circ n}\right)^{-1}(V)\right)_{n \geq 0}$ is a decreasing sequence of disks and the map $f \colon \left(f^{\circ(n+1)}\right)^{-1}(V) \to \left(f^{\circ n}\right)^{-1}(V)$ has degree e for each $n \geq 0$. For $n \geq 0$, denote by $r_n \in \mathbb{R}_{>0}$ the radius of $\left(f^{\circ n}\right)^{-1}(V)$. We have $\frac{r_{n-1}}{r_n} = \left(\frac{r_n}{r_{n+1}}\right)^e$ for each $n \geq 1$ by Lemma 60. Now, suppose that $z_0 \in K$ is a periodic point for f with period $p \geq 1$. As $z_0 \in \bigcap_{n \geq 0} \left(f^{\circ n}\right)^{-1}(V)$, we have $\left|\left(f^{\circ p}\right)'(z_0)\right| \leq \frac{r_n}{r_{n+p}}$ for each $n \geq 0$ by Lemma 46. Thus, we have $\left|\left(f^{\circ p}\right)'(z_0)\right| \leq \left(\frac{r_0}{r_p}\right)^{\frac{1}{e^n}}$ for each $n \geq 0$, which yields $\left|\left(f^{\circ p}\right)'(z_0)\right| \leq 1$ by letting $n \to +\infty$. This completes the proof of the lemma. $\square$

Exhibiting explicit examples, we obtain the two results below, which show that the bounds in Theorem B are optimal in the non-Archimedean case.

Proposition 65. *Assume here that $d \geq 4$. Then, for every $R \in |K^*|$, there exists $f \in \mathrm{Poly}_d(K)$ such that*

$$M_f = \log^+(R)\,, \quad M_f^{(1)} = 0 \quad \text{and} \quad M_f^{(2)} = C_d \cdot M_f\,,$$

where $C_d \in \mathbb{R}_{>0}$ is defined in Theorem B.

Proof. Observe that, if $R \in (0, 1]$, then $f(z) = z^d$ satisfies the required conditions. Now, assume that $R \in |K^*| \cap (1, +\infty)$. Choose $\gamma \in K^*$ such that $|\gamma| = R$. Set

$$(d_0, d_1) = \begin{cases} \left(\frac{d}{2}, \frac{d}{2}\right) & \text{if } d \text{ is even} \\ \left(\frac{d-1}{2}, \frac{d+1}{2}\right) & \text{if } d \text{ is odd} \end{cases}, \quad \text{so that} \quad C_d = \frac{d-1}{2}\left(\frac{1}{d_0} + \frac{1}{d_1}\right).$$

Define

$$f(z) = f_{\boldsymbol{c}}(z) = \sum_{j=0}^{d_1 - 1} b_j z^{d_0 + j} (z - \gamma)^{d_1 - 1 - j} \left(d_0 z - (d_0 + 1 + j)\,\omega\right),$$

with $b_j = \frac{(-1)^j (d_0 - 1)! (d_1 - 1)!}{(d_0 + 1 + j)! (d_1 - 1 - j)!}$ for all $j \in \{0, \dots, d_1 - 1\}$, where

$$\omega = \frac{d_0}{d}\gamma + \frac{(-1)^{d_1}(d-1)!}{(d_0 - 1)!\,(d_1 - 1)!}\gamma^{2-d} \quad \text{and} \quad \boldsymbol{c} = \left(\underbrace{0, \dots, 0}_{d_0 - 1 \text{ entries}}, \underbrace{\gamma, \dots, \gamma}_{d_1 - 1 \text{ entries}}, \omega\right).$$

Thus, we have

$$f(0) = 0\,, \quad f(\gamma) = \gamma \quad \text{and} \quad f'(z) = z^{d_0 - 1}(z - \gamma)^{d_1 - 1}(z - \omega)\,.$$

Note that $|\omega| = R$. As a result, we have $M_f = \log(R) > 0$ by Claim 53. Moreover, we have $g_f(0) = 0$ and $g_f(\gamma) = 0$, which yields $g_f(\omega) = M_f$. Therefore, $\{g_f < M_f\}$

is the union of two disjoint open disks V_0, V_1 of radius R by Lemma 55. Without loss of generality, we can assume that $0 \in V_0$ and $\gamma \in V_1$. Thus, by Lemmas 47, 49 and 54, the maps $f_0 \colon V_0 \to \{g_f < d \cdot M_f\}$ and $f_1 \colon V_1 \to \{g_f < d \cdot M_f\}$ induced by f have degrees d_0 and d_1, respectively, since 0 and γ are critical points for f with multiplicities $d_0 - 1$ and $d_1 - 1$, respectively. Now, by Lemmas 47 and 49, $f_0^{-1}(V_1)$ is the union of d_0 pairwise distinct disk components $U_0^{(1)}, \dots, U_0^{(d_0)}$ of $\left\{g_f < \frac{M_f}{d}\right\}$ and f maps bijectively $U_0^{(k)}$ to V_1 for all $k \in \{1, \dots, d_0\}$. Similarly, $f_1^{-1}(V_0)$ is the union of d_1 pairwise distinct disk components $U_1^{(1)}, \dots, U_1^{(d_1)}$ of $\left\{g_f < \frac{M_f}{d}\right\}$ and f maps bijectively $U_1^{(\ell)}$ to V_0 for all $\ell \in \{1, \dots, d_1\}$.

Now, suppose that $k \in \{1, \dots, d_0\}$ and $\ell \in \{1, \dots, d_1\}$. Then there is a unique periodic point $z_0 \in K$ for f with period 2 such that $z_0 \in U_0^{(k)}$ and $f(z_0) \in U_1^{(\ell)}$ by Lemma 61. Now, define $\overline{U}_0 = f_0^{-1}(\{g_f \leq M_f\})$ and $\overline{U}_1 = f_1^{-1}(\{g_f \leq M_f\})$. Then $\overline{U}_0$ and $\overline{U}_1$ are disks and the maps $f \colon \overline{U}_0 \to \{g_f \leq M_f\}$ and $f \colon \overline{U}_1 \to \{g_f \leq M_f\}$ have degrees d_0 and d_1, respectively, by Lemmas 47, 49 and 54. Thus, $\overline{U}_0$ and $\overline{U}_1$ are also the disk components of $\left\{g_f \leq \frac{M_f}{d}\right\}$ containing z_0 and $f(z_0)$, respectively. Therefore, by Lemma 61, we have

$$\frac{1}{2} \log \left| \left(f^{\circ 2}\right)'(z_0) \right| = \frac{d-1}{2} \left(\frac{1}{d_0} + \frac{1}{d_1} \right) M_f = C_d \cdot M_f \,.$$

Let us conclude the proof of the proposition. If $z_0 \in K$ is any fixed point for f, then the point z_0 is also fixed for $f_j \colon V_j \to \{g_f < d \cdot M_f\}$ for some $j \in \{0, 1\}$, and hence $\log |f'(z_0)| \leq 0$ by Lemma 64. In addition, we always have $M_f^{(1)} \geq 0$ by the holomorphic fixed-point formula, as shown in the proof of Theorem B. This shows that $M_f^{(1)} = 0$. Now, suppose that $z_0 \in K$ is a periodic point for f with period 2. If z_0 and $f(z_0)$ both lie in V_j for some $j \in \{0, 1\}$, then the point z_0 is also periodic for f_j, and therefore $\frac{1}{2} \log \left| \left(f^{\circ 2}\right)'(z_0) \right| \leq 0$ by Lemma 64. Otherwise, replacing z_0 by $f(z_0)$ if necessary, we have $z_0 \in U_0^{(k)}$ and $f(z_0) \in U_1^{(\ell)}$ for some $k \in \{1, \dots, d_0\}$ and some $\ell \in \{1, \dots, d_1\}$, and hence $\frac{1}{2} \log \left| \left(f^{\circ 2}\right)'(z_0) \right| = C_d \cdot M_f$ by the discussion above. Moreover, we also proved that the latter case occurs for some choices of z_0. Thus, we have $M_f^{(2)} = C_d \cdot M_f$. This completes the proof of the proposition. □

Proposition 66. *Assume here that $d \geq 4$. Then, for every $R \in |K^*|$, there exists $f \in \mathrm{Poly}_d(K)$ such that*

$$M_f = \log^+(R) \,, \quad M_f^{(1)} = \frac{d-1}{d-2} M_f \quad \textit{and} \quad M_f^{(2)} = \frac{d-1}{d-2} M_f \,.$$

Proof. Observe that, if $R \in (0, 1]$, then $f(z) = z^d$ satisfies the required conditions. Now, assume that $R \in |K^*| \cap (1, +\infty)$. Choose $\gamma \in K^*$ such that $|\gamma| = R$. Define

$$f(z) = f_{\boldsymbol{c}}(z) = \frac{1}{d} z^2 (z - \gamma)^{d-2} \,, \quad \text{with} \quad \boldsymbol{c} = \left(0, \underbrace{\gamma, \dots, \gamma}_{d-3 \text{ entries}}, \frac{2}{d}\gamma \right) \in K^{d-1} \,.$$

Thus, we have

$$f(0) = 0 \,, \quad f(\gamma) = 0 \quad \text{and} \quad f'(z) = z (z - \gamma)^{d-3} \left(z - \frac{2}{d}\gamma \right) \,.$$

Note that $M_f = \log(R) > 0$ by Claim 53. Moreover, $g_f(0) = 0$ and $g_f(\gamma) = 0$, and hence $g_f\left(\frac{2}{d}\gamma\right) = M_f$. Therefore, $\{g_f < M_f\}$ is the union of two disjoint open disks V_0, V_1 of radius R by Lemma 55. Without loss of generality, we may assume that $0 \in V_0$ and $\gamma \in V_1$. Since 0 and γ are critical points for f with multiplicities 1 and $d-3$, respectively, the maps $f_0\colon V_0 \to \{g_f < d \cdot M_f\}$ and $f_1\colon V_1 \to \{g_f < d \cdot M_f\}$ induced by f have degrees 2 and $d-2$, respectively, by Lemmas 47, 49 and 54. By similar arguments, the sets $\overline{U}_0 = f_0^{-1}\left(\{g_f \leq M_f\}\right)$ and $\overline{U}_1 = f_1^{-1}\left(\{g_f \leq M_f\}\right)$ are all the disk components of $\left\{g_f \leq \frac{M_f}{d}\right\}$ and the induced maps $f\colon \overline{U}_0 \to \{g_f \leq M_f\}$ and $f\colon \overline{U}_1 \to \{g_f \leq M_f\}$ have degrees 2 and $d-2$, respectively. The set $f_1^{-1}(V_1)$ is the union of $d-2$ distinct disk components $U_1^{(1)}, \dots, U_1^{(d-2)}$ of $\left\{g_f < \frac{M_f}{d}\right\}$ and f maps bijectively $U_1^{(k)}$ onto V_1 for each $k \in \{1, \dots, d-2\}$ by Lemmas 47 and 49. Similarly, the set $f_0^{-1}(V_1)$ is the union of two distinct disk components $W_0^{(1)}, W_0^{(2)}$ of $\left\{g_f < \frac{M_f}{d}\right\}$ and f maps bijectively $W_0^{(k)}$ onto V_1 for each $k \in \{1, 2\}$. Moreover, the set $W_1 = f_1^{-1}(V_0)$ is a disk component of $\left\{g_f < \frac{M_f}{d}\right\}$ and f maps W_1 onto V_0 with degree $d-2$.

Suppose that $k \in \{1, \dots, d-2\}$. Then, by the discussion above and Lemma 61, there is a unique fixed point $z_0 \in U_1^{(k)}$ for f and we have $\log\left|f'(z_0)\right| = \frac{d-1}{d-2} M_f$.

Now, suppose that $k, \ell \in \{1, \dots, d-2\}$ are distinct. Then, by Lemma 61, there exists a unique periodic point $z_0 \in K$ for f with period 2 such that $z_0 \in U_1^{(k)}$ and $f(z_0) \in U_1^{(\ell)}$ and we have $\frac{1}{2}\log\left|\left(f^{\circ 2}\right)'(z_0)\right| = \frac{d-1}{d-2} M_f$.

Finally, suppose that $z_0 \in K$ is any periodic point for f with period 2 such that $z_0 \in V_0$ and $f(z_0) \in V_1$. Note that $z_0 \in W_0^{(k)}$ and $f(z_0) \in W_1$ for some $k \in \{1, 2\}$. Now, denote by D_0 and D_1 the disk components of $\left\{g_f < \frac{M_f}{d^2}\right\}$ containing z_0 and $f(z_0)$, respectively. Then f induces bijections from D_0 onto W_1 and from D_1 onto $W_0^{(k)}$ by Lemmas 47 and 49. Therefore, z_0 is the unique periodic point for f with period 2 such that $z_0 \in D_0$ and $f(z_0) \in D_1$ by Lemma 61. First, observe that the maps $f^{\circ 2}\colon W_0^{(k)} \to \{g_f < d \cdot M_f\}$ and $f^{\circ 2}\colon W_1 \to \{g_f < d \cdot M_f\}$ have degrees $d-2$ and $2(d-2)$, respectively. Next, denote here by $\overline{D}_0$ and $\overline{D}_1$ the disk components of $\left\{g_f \leq \frac{M_f}{d^2}\right\}$ containing z_0 and $f(z_0)$, respectively. Then, by Lemmas 47 and 49, f maps bijectively $\overline{D}_0$ onto $\overline{U}_1$, we have $\overline{D}_1 = f_1^{-1}\left(\overline{U}_0\right)$ and the map $f\colon \overline{D}_1 \to \overline{U}_0$ has degree $d-2$. It follows that $f^{\circ 2}\colon \overline{D}_0 \to \{g_f \leq M_f\}$ and $f^{\circ 2}\colon \overline{D}_1 \to \{g_f \leq M_f\}$ have degrees $d-2$ and $2(d-2)$, respectively. Therefore, by Lemma 61, we have

$$\frac{1}{2}\log\left|\left(f^{\circ 2}\right)'(z_0)\right| = \frac{d-1}{2}\left(\frac{1}{d-2} + \frac{1}{2(d-2)}\right) M_f = \frac{3(d-1)}{4(d-2)} M_f\,.$$

Let us conclude the proof of the proposition. Suppose that $z_0 \in K$ is any fixed point for f. If $z_0 \in V_0$, then the point z_0 is also fixed for $f_0\colon V_0 \to \{g_f < d \cdot M_f\}$, which yields $\log\left|f'(z_0)\right| \leq 0$ by Lemma 64. Otherwise, we have $z_0 \in U_1^{(k)}$ for some $k \in \{1, \dots, d-2\}$, and hence $\log\left|f'(z_0)\right| = \frac{d-1}{d-2} M_f$ by the previous discussion. In addition, we also proved that the latter case occurs for certain choices of z_0. This shows that $M_f^{(1)} = \frac{d-1}{d-2} M_f$. Now, suppose that $z_0 \in K$ is any periodic point for f with period 2. If z_0 and $f(z_0)$ both lie in V_0, then the point z_0 is also periodic for f_0, and hence $\frac{1}{2}\log\left|\left(f^{\circ 2}\right)'(z_0)\right| \leq 0$ by Lemma 64. If z_0 and $f(z_0)$ both lie in V_1, then $z_0 \in U_1^{(k)}$ and $f(z_0) \in U_1^{(\ell)}$ for some distinct $k, \ell \in \{1, \dots, d-2\}$, and hence $\frac{1}{2} \log \left| \left(f^{\circ 2}\right)'(z_0) \right| = \frac{d-1}{d-2} M_f$ by the previous discussion. Otherwise, replacing z_0 by $f(z_0)$ if necessary, we have $z_0 \in V_0$ and $f(z_0) \in V_1$, and thus the discussion above yields $\frac{1}{2} \log \left| \left(f^{\circ 2}\right)'(z_0) \right| = \frac{3(d-1)}{4(d-2)} M_f$. Furthermore, we also proved that the second case occurs for some choices of z_0. Thus, we have $M_f^{(2)} = \frac{d-1}{d-2} M_f$. This completes the proof of the proposition. □

Applying Propositions 65 and 66 with the non-Archimedean field $K = \mathbb{C}\left\{\left\{\frac{1}{t}\right\}\right\}$ of convergent complex Puiseux series in $\frac{1}{t}$, with t an indeterminate, we shall show that the bounds in Theorem B are also optimal in the complex setting.

Assume here that t is an indeterminate and $K = \mathbb{C}\left\{\left\{\frac{1}{t}\right\}\right\}$ is the field of Puiseux series in $\frac{1}{t}$ with coefficients in $\mathbb{C}$ that converge on some punctured neighborhood of $t = \infty$. Then K is algebraically closed according to the Newton–Puiseux theorem (see [Now00]). We equip K with its usual absolute value $|.|$, which is given by

$$|a| = \lim_{t \to \infty} \exp\left(\frac{\log |a(t)|_\infty}{\log |t|_\infty}\right),$$

where $|.|_\infty$ denotes the usual absolute value on $\mathbb{C}$. Thus, K is a non-Archimedean valued field and its residue field is naturally isomorphic to $\mathbb{C}$. Now, note that any meromorphic function on a neighborhood of $t = \infty$ in $\widehat{\mathbb{C}}$ can be identified with an element of K via its Laurent series expansion at $t = \infty$. In particular, denoting by $\overline{\mathbb{D}}$ the closed unit disk around the origin in $\mathbb{C}$, every holomorphic family $(f_t)_{t \in \mathbb{C} \setminus \overline{\mathbb{D}}}$ of elements of $\operatorname{Poly}_d(\mathbb{C})$ with a pole at $t = \infty$ induces some element $f \in \operatorname{Poly}_d(K)$. Specifically, if a holomorphic family $(f_t)_{t \in \mathbb{C} \setminus \overline{\mathbb{D}}}$ of elements of $\operatorname{Poly}_d(\mathbb{C})$ with a pole at $t = \infty$ is given by $f_t(z) = \sum_{j=0}^{d} a_j(t) z^j$, with $a_0, \dots, a_d$ holomorphic on $\mathbb{C} \setminus \overline{\mathbb{D}}$ and meromorphic on $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, then the induced element $f \in \operatorname{Poly}_d(K)$ is $f(z) = \sum_{j=0}^{d} a_j z^j$.

To deduce results in the complex setting from analogous statements in the non-Archimedean case, we shall use the result below. It is a particular case of a result due to DeMarco.

Lemma 67 ([DeM16, Proposition 3.1])**.** *Suppose here that $K = \mathbb{C}\left\{\left\{\frac{1}{t}\right\}\right\}$ is the field of convergent complex Puiseux series in $\frac{1}{t}$, $(f_t)_{t \in \mathbb{C} \setminus \overline{\mathbb{D}}}$ is a holomorphic family of elements of $\operatorname{Poly}_d(\mathbb{C})$ that has a pole at $t = \infty$ and $f \in \operatorname{Poly}_d(K)$ is the element induced by $(f_t)_{t \in \mathbb{C} \setminus \overline{\mathbb{D}}}$. Then $M_{f_t} = M_f \cdot \log |t|_\infty + o(\log |t|_\infty)$ as $t \to \infty$.*

We shall also use the following fact:

Lemma 68. *Suppose here that $K = \mathbb{C}\left\{\left\{\frac{1}{t}\right\}\right\}$ is the field of convergent complex Puiseux series in $\frac{1}{t}$, $(f_t)_{t \in \mathbb{C} \setminus \overline{\mathbb{D}}}$ is a holomorphic family of elements of $\operatorname{Poly}_d(\mathbb{C})$ that has a pole at $t = \infty$ and $f \in \operatorname{Poly}_d(K)$ is the element induced by $(f_t)_{t \in \mathbb{C} \setminus \overline{\mathbb{D}}}$. Then, for every integer $p \geq 1$, we have $M_{f_t}^{(p)} = M_f^{(p)} \cdot \log |t|_\infty + O(1)$ as $t \to \infty$.*

Proof. Suppose that $p \geq 1$ is an integer. Write

$$\Lambda_f^{(p)} = [\lambda_1, \dots, \lambda_N] \in K^N / \mathfrak{S}_N, \quad \text{with} \quad N = N_d^{(p)}.$$

There exists $R \in \mathbb{R}_{>0}$ such that the complex Puiseux series $\lambda_1, \dots, \lambda_N$ all converge on $\mathbb{C} \setminus \overline{\mathbb{D}}_R$, where $\overline{\mathbb{D}}_R$ denotes the closed disk of center 0 and radius R in $\mathbb{C}$. Then, as the elementary symmetric functions of the multipliers of polynomials of degree d at all their cycles with period p define regular functions $\sigma_{d,j}^{(p)} \circ \pi_d$ on Poly_d, with $j \in \{1, \ldots, N\}$, we have

$$\forall t \in \mathbb{C} \setminus \overline{\mathbb{D}}_R,\ \Lambda_{f_t}^{(p)} = [\lambda_1(t), \ldots, \lambda_N(t)] \in \mathbb{C}^N / \mathfrak{S}_N\,.$$

Finally, note that $\log|a(t)|_\infty = \log|a| \cdot \log|t|_\infty + O(1)$ as $t \to \infty$ for each $a \in K$. In particular, for each $j \in \{1, \ldots, N\}$, we have $\log|\lambda_j(t)|_\infty = \log|\lambda_j| \cdot \log|t|_\infty + O(1)$ as $t \to \infty$. This completes the proof of the lemma. □

Finally, combining Propositions 65 and 66 with Lemmas 67 and 68, we directly deduce the two results below, which show that the bounds in Theorem B are also sharp in the complex case.

Corollary 69. *Assume that $d \geq 4$. Then there exists a rational family $(f_t)_{t \in \mathbb{C}^*}$ of elements of $\mathrm{Poly}_d(\mathbb{C})$ such that f_t degenerates in $\mathcal{P}_d(\mathbb{C})$ as $t \to \infty$ and*

$$M_{f_t}^{(1)} = O(1) \quad \textit{and} \quad M_{f_t}^{(2)} \sim C_d \cdot M_{f_t} \quad \textit{as} \quad t \to \infty\,.$$

Proof. Consider the rational family $(f_t)_{t \in \mathbb{C}^*}$ of elements of $\mathrm{Poly}_d(\mathbb{C})$ defined by

$$f_t(z) = \sum_{j=0}^{d_1-1} b_j z^{d_0+j} (z-t)^{d_1-1-j} \left(d_0 z - (d_0+1+j)\,\omega_t\right),$$

with $b_j = \frac{(-1)^j (d_0-1)!(d_1-1)!}{(d_0+1+j)!(d_1-1-j)!}$ for all $j \in \{0, \ldots, d_1-1\}$, where

$$(d_0, d_1) = \begin{cases} \left(\frac{d}{2}, \frac{d}{2}\right) & \text{if } d \text{ is even} \\ \left(\frac{d-1}{2}, \frac{d+1}{2}\right) & \text{if } d \text{ is odd} \end{cases} \quad \text{and} \quad \omega_t = \frac{d_0}{d} t + \frac{(-1)^{d_1}(d-1)!}{(d_0-1)!\,(d_1-1)!} t^{2-d}\,.$$

Now, assume that $K = \mathbb{C}\left\{\left\{\frac{1}{t}\right\}\right\}$, and denote by $f \in \mathrm{Poly}_d(K)$ the element induced by $(f_t)_{t \in \mathbb{C}^*}$. Then, as $|t| = \exp(1)$, the proof of Proposition 65 shows that $M_f = 1$, $M_f^{(1)} = 0$ and $M_f^{(2)} = C_d \cdot M_f$. Thus, the desired result follows immediately from Lemmas 67 and 68. □

Corollary 70. *Assume that $d \geq 4$. Then there exists a polynomial family $(f_t)_{t \in \mathbb{C}}$ of elements of $\mathrm{Poly}_d(\mathbb{C})$ such that f_t degenerates in $\mathcal{P}_d(\mathbb{C})$ as $t \to \infty$ and*

$$M_{f_t}^{(1)} \sim \frac{d-1}{d-2} M_{f_t} \quad \textit{and} \quad M_{f_t}^{(2)} \sim \frac{d-1}{d-2} M_{f_t} \quad \textit{as} \quad t \to \infty\,.$$

Proof. Consider the polynomial family $(f_t)_{t \in \mathbb{C}}$ of elements of $\mathrm{Poly}_d(\mathbb{C})$ defined by

$$f_t(z) = \frac{1}{d} z^2 (z-t)^{d-2}\,.$$

Now, assume that $K = \mathbb{C}\left\{\left\{\frac{1}{t}\right\}\right\}$, and denote by $f \in \mathrm{Poly}_d(K)$ the element induced by $(f_t)_{t \in \mathbb{C}}$. Then, as $|t| = \exp(1)$, the proof of Proposition 66 shows that $M_f = 1$, $M_f^{(1)} = \frac{d-1}{d-2} M_f$ and $M_f^{(2)} = \frac{d-1}{d-2} M_f$. Thus, the desired result follows immediately from Lemmas 67 and 68. □

5. Unique determination of a generic conjugacy class of polynomial maps by its multipliers at its small cycles

In this section, we shall prove Theorem C. As before, we fix an integer $d \geq 2$.

5.1. **Some preliminaries.** First, let us present the ingredients that we use in our proof of Theorem C.

Since $\mathcal{P}_d(\mathbb{C}) \cong \mathcal{P}_d^{\mathrm{mc}}(\mathbb{C})$, we can restrict our attention to monic centered complex polynomials. Recall here that

$$\mathrm{Poly}_d^{\mathrm{mc}}(\mathbb{C}) = \left\{ z^d + \sum_{j=0}^{d-2} b_j z^j : b_0, \dots, b_{d-2} \in \mathbb{C} \right\} .$$

Now, define $\alpha = \exp\left(\frac{2\pi i}{d-1}\right)$, so that

$$\mu_{d-1}(\mathbb{C}) = \langle \alpha \rangle = \left\{ \alpha^k : k \in \{0, \dots, d-2\} \right\} .$$

Also recall that the group $\mu_{d-1}(\mathbb{C})$ acts on $\mathrm{Poly}_d^{\mathrm{mc}}(\mathbb{C})$ via $\omega \centerdot f = \omega f\left(\frac{z}{\omega}\right)$ and that $\mathcal{P}_d(\mathbb{C})$ is biholomorphic to the quotient $\mathcal{P}_d^{\mathrm{mc}}(\mathbb{C})$ of $\mathrm{Poly}_d^{\mathrm{mc}}(\mathbb{C})$ by $\mu_{d-1}(\mathbb{C})$.

Our objective is to prove the result below, which directly implies Theorem C.

Lemma 71. *There exists a nonempty open subset U of $\mathrm{Poly}_d^{mc}(\mathbb{C})$ such that, for every $f \in U$,*

$$\left\{ g \in \mathrm{Poly}_d^{mc}(\mathbb{C}) : \Lambda_g^{(1)} = \Lambda_f^{(1)} \text{ and } \Lambda_g^{(2)} = \Lambda_f^{(2)} \right\} = \left\{ \alpha^k \centerdot f : k \in \{0, \dots, d-2\} \right\} .$$

Define Ξ to be the set of all elements $[\lambda_0, \dots, \lambda_{d-1}] \in \mathbb{C}^d / \mathfrak{S}_d$ that satisfy $\lambda_j \neq 1$ for all $j \in \{0, \dots, d-1\}$ and

$$\forall J \subseteq \{0, \dots, d-1\},\ \sum_{j \in J} \frac{1}{1 - \lambda_j} = 0 \Longleftrightarrow J = \varnothing \text{ or } \{0, \dots, d-1\} .$$

To prove Lemma 71, we shall use the result below, which is due to Fujimura.

Lemma 72 ([Fuj07, Theorem 6])**.** *Suppose that $\Lambda \in \Xi$. Then there exist at most $(d-1)!$ elements $f \in \mathrm{Poly}_d^{mc}(\mathbb{C})$ such that $\Lambda_f^{(1)} = \Lambda$.*

Now, define

$$f_0(z) = z^d \in \mathrm{Poly}_d^{\mathrm{mc}}(\mathbb{C}) .$$

We shall also use the explicit expressions for the differentials of multiplier maps at f_0, which are due to Gorbovickis.

Lemma 73 ([Gor16, Lemma 3.1])**.** *Suppose that $z_0 \in \mathbb{C}^*$ is a periodic point for f_0 with period $p \geq 1$, U is an open neighborhood of f_0 in $\mathrm{Poly}_d^{mc}(\mathbb{C})$, $\zeta_0 \colon U \to \mathbb{C}$ is a holomorphic map such that $\zeta_0(f_0) = z_0$ and $f^{\circ p}(\zeta_0(f)) = \zeta_0(f)$ for all $f \in U$ and $\rho_0 \colon U \to \mathbb{C}$ is the holomorphic map defined by $\rho_0(f) = (f^{\circ p})'(\zeta_0(f))$. Then*

$$\forall k \in \{0, \dots, d-2\},\ \frac{\partial \rho_0}{\partial a_k}(f_0) = d^{p-1}(k-d) \sum_{j=0}^{p-1} z_0^{d^j (k-d)} .$$

5.2. **Multipliers at fixed points.** Here, let us parametrize some open neighborhood U_1 of f_0 in $\mathrm{Poly}_d^{\mathrm{mc}}(\mathbb{C})$ by the multipliers at the fixed points and describe, for a generic $f \in U_1$, all the elements $g \in \mathrm{Poly}_d^{\mathrm{mc}}(\mathbb{C})$ such that $\Lambda_g^{(1)} = \Lambda_f^{(1)}$.

The fixed points for f_0 are precisely 0 and the points α^j, with $j \in \{0, \dots, d-2\}$. In addition, we have $f_0'(0) = 0$ and $f_0'\left(\alpha^j\right) = d$ for all $j \in \{0, \dots, d-2\}$. It follows from the implicit function theorem that there exist an open neighborhood U_1 of f_0

in $\mathrm{Poly}_d^{\mathrm{mc}}(\mathbb{C})$ and holomorphic maps $\zeta_j^{(1)} \colon U_1 \to \mathbb{C}$, with $j \in \{0, \dots, d-2\} \cup \{\lozenge\}$, such that

$$\zeta_j^{(1)}(f_0) = \begin{cases} \alpha^j & \text{if } j \in \{0, \dots, d-2\} \\ 0 & \text{if } j = \lozenge \end{cases} \quad \text{and} \quad \forall f \in U_1,\ f\left(\zeta_j^{(1)}(f)\right) = \zeta_j^{(1)}(f)$$

for all $j \in \{0, \dots, d-2\} \cup \{\lozenge\}$. Shrinking U_1 if necessary, we may assume that the points $\zeta_j^{(1)}(f)$, with $j \in \{0, \dots, d-2\} \cup \{\lozenge\}$, are pairwise distinct for each $f \in U_1$. For every $f \in U_1$, we have

$$\Phi_f^{(1)}(z) = \left(z - \zeta_\lozenge^{(1)}(f)\right) \prod_{j=0}^{d-2} \left(z - \zeta_j^{(1)}(f)\right) .$$

For $j \in \{0, \dots, d-2\} \cup \{\lozenge\}$, define the holomorphic map $\rho_j^{(1)} \colon U_1 \to \mathbb{C} \setminus \{1\}$ by

$$\rho_j^{(1)}(f) = f'\left(\zeta_j^{(1)}(f)\right) .$$

For every $f \in U_1$, we have

$$\Lambda_f^{(1)} = \left[\rho_0^{(1)}(f), \dots, \rho_{d-2}^{(1)}(f), \rho_\lozenge^{(1)}(f)\right] \quad \text{and} \quad \frac{1}{1 - \rho_\lozenge^{(1)}(f)} + \sum_{j=0}^{d-2} \frac{1}{1 - \rho_j^{(1)}(f)} = 0 .$$

Now, define the holomorphic map

$$\boldsymbol{\rho}_1 = \left(\rho_0^{(1)}, \dots, \rho_{d-2}^{(1)}\right) : U_1 \to \mathbb{C}^{d-1} .$$

Denote by $\cdot$ the natural action of $\mathfrak{S}_{d-1}$ on $\mathbb{C}^{d-1}$, which is given by

$$\sigma \cdot (\lambda_0, \dots, \lambda_{d-2}) = \left(\lambda_{\sigma^{-1}(0)}, \dots, \lambda_{\sigma^{-1}(d-2)}\right) .$$

By the formulas above,

$$\forall f, g \in U_1,\ (\exists \sigma \in \mathfrak{S}_{d-1},\ \boldsymbol{\rho}_1(g) = \sigma \cdot \boldsymbol{\rho}_1(f)) \Longrightarrow \Lambda_f^{(1)} = \Lambda_g^{(1)} . \tag{5.1}$$

Now, define

$$A_1 = \left(\frac{\partial \rho_j^{(1)}}{\partial a_k}(f_0)\right)_{0 \leq j,k \leq d-2} \in M_{(d-1)\times(d-1)}(\mathbb{C})$$

to be the Jacobian matrix of $\boldsymbol{\rho}_1$ at f_0. By Lemma 73, we have

$$A_1 = \left((k-d)\alpha^{j(k-1)}\right)_{0 \leq j,k \leq d-2} .$$

As a consequence, we have the key result below. For $k \in \{0, \dots, d-2\}$, define the polynomial

$$P_k(T) = \alpha^k \prod_{\substack{0 \leq j \leq d-2 \\ j \neq k}} \left(\frac{T - \alpha^j}{\alpha^k - \alpha^j}\right) \in \mathbb{C}[T] .$$

Claim 74. The matrix A_1 is invertible and $A_1^{-1} = (B_{j,k})_{0 \leq j,k \leq d-2}$, where

$$\forall k \in \{0, \dots, d-2\},\ P_k(T) = \sum_{j=0}^{d-2} (j-d) B_{j,k} T^j .$$

Proof. Suppose that

$$B = (B_{j,k})_{0 \leq j,k \leq d-2} \in M_{(d-1)\times(d-1)}(\mathbb{C})\,.$$

For $k \in \{0, \ldots, d-2\}$, define the polynomial

$$Q_k(T) = \sum_{j=0}^{d-2} (j-d) B_{j,k} T^j \in \mathbb{C}[T]\,.$$

Also denote by δ the Kronecker delta. Then

$$\begin{aligned} A_1 B = I_{d-2} &\Longleftrightarrow \forall j,k \in \{0,\ldots,d-2\},\ \sum_{\ell=0}^{d-2} (\ell-d) B_{\ell,k} \alpha^{j(\ell-1)} = \delta_{jk} \\ &\Longleftrightarrow \forall j,k \in \{0,\ldots,d-2\},\ Q_k\left(\alpha^j\right) = \alpha^j \delta_{jk} \\ &\Longleftrightarrow \forall k \in \{0,\ldots,d-2\},\ Q_k = P_k\,. \end{aligned}$$

This completes the proof of the claim. ☐

By Claim 74 and the inverse function theorem, shrinking U_1 if necessary, we can assume that $\boldsymbol{\rho}_1$ induces a biholomorphism from U_1 onto an open neighborhood V_1 of $(d, \ldots, d)$ in $\mathbb{C}^{d-1}$. In addition, shrinking further U_1 if necessary, we can assume that U_1 is connected, $\Lambda_f^{(1)} \in \Xi$ for all $f \in U_1$ and V_1 is invariant under the natural action of $\mathfrak{S}_{d-1}$.

Now, define the action $*$ of $\mathfrak{S}_{d-1}$ on U_1 by

$$\sigma * f = \boldsymbol{\rho}_1^{-1}\left(\sigma \cdot \boldsymbol{\rho}_1(f)\right)\,.$$

Denote by Δ the fat diagonal of $\mathbb{C}^{d-1}$, which is given by

$$\Delta = \bigcup_{0 \leq j < k \leq d-2} \left\{(\lambda_0, \ldots, \lambda_{d-2}) \in \mathbb{C}^{d-1} : \lambda_j = \lambda_k\right\}\,.$$

By (5.1), we have $\Lambda_{\sigma * f}^{(1)} = \Lambda_f^{(1)} \in \Xi$ for all $f \in U_1$ and all $\sigma \in \mathfrak{S}_{d-1}$. Moreover, for each $f \in U_1 \setminus \boldsymbol{\rho}_1^{-1}(\Delta)$, the elements $\sigma * f$, with $\sigma \in \mathfrak{S}_{d-1}$, are pairwise distinct. It follows from Lemma 72 that

$$\forall f \in U_1 \setminus \boldsymbol{\rho}_1^{-1}(\Delta),\ \left\{g \in \operatorname{Poly}_d^{\mathrm{mc}}(\mathbb{C}) : \Lambda_g^{(1)} = \Lambda_f^{(1)}\right\} = \left\{\sigma * f : \sigma \in \mathfrak{S}_{d-1}\right\}\,. \tag{5.2}$$

In addition, we can describe conjugation in terms of this action. Define the cyclic permutation

$$\sigma_0 = (0\ \ldots\ d-2) \in \mathfrak{S}_{d-1}\,.$$

Claim 75. We have $\alpha^k \centerdot f = \sigma_0^k * f$ for all $f \in U_1$ and all $k \in \{0, \ldots, d-2\}$.

Proof. The set $U_1 \setminus \boldsymbol{\rho}_1^{-1}(\Delta)$ is a connected open subset of $\operatorname{Poly}_d^{\mathrm{mc}}(\mathbb{C})$. Moreover, if $f \in U_1 \setminus \boldsymbol{\rho}_1^{-1}(\Delta)$ and $k \in \{0, \ldots, d-2\}$, then there exists a permutation $\sigma \in \mathfrak{S}_{d-1}$ such that $\alpha^k \centerdot f = \sigma * f$ by (5.2), as $\Lambda_{\alpha^k \centerdot f}^{(1)} = \Lambda_f^{(1)}$, and hence $\alpha^k \centerdot f \in U_1 \setminus \boldsymbol{\rho}_1^{-1}(\Delta)$. Thus, $U_1 \setminus \boldsymbol{\rho}_1^{-1}(\Delta)$ is invariant under the action of $\mu_{d-1}(\mathbb{C}) = \langle \alpha \rangle$ by conjugation. As a result, if $j, k \in \{0, \ldots, d-2\}$, then there exists $\ell \in \{0, \ldots, d-2\} \cup \{\Diamond\}$ such that $\zeta_j^{(1)}\left(\alpha^k \centerdot f\right) = \alpha^k \zeta_\ell^{(1)}(f)$ for all $f \in U_1 \setminus \boldsymbol{\rho}_1^{-1}(\Delta)$, and we obtain $\ell = \sigma_0^{-k}(j)$ by letting $f \to f_0$. Thus, we have $\zeta_j^{(1)}\left(\alpha^k \centerdot f\right) = \alpha^k \zeta_{\sigma_0^{-k}(j)}^{(1)}(f)$ for all $f \in U_1 \setminus \boldsymbol{\rho}_1^{-1}(\Delta)$ and all $j, k \in \{0, \ldots, d-2\}$. As a consequence, for each $f \in U_1 \setminus \boldsymbol{\rho}_1^{-1}(\Delta)$ and each $k \in \{0, \ldots, d-2\}$, we have $\boldsymbol{\rho}_1\left(\alpha^k \centerdot f\right) = \sigma_0^k \cdot \boldsymbol{\rho}_1(f)$ since the multiplier is invariant under conjugation, which yields $\alpha^k \centerdot f = \sigma_0^k * f$. Thus, $f \mapsto \alpha^k \centerdot f$ and $f \mapsto \sigma_0^k * f$ coincide on $U_1 \setminus \boldsymbol{\rho}_1^{-1}(\Delta)$, and hence they coincide on all of U_1 since $U_1 \setminus \boldsymbol{\rho}_1^{-1}(\Delta)$ is dense in U_1. This completes the proof of the claim. □

5.3. **Multipliers at cycles with period 2.** Here, let us examine the variations of the multipliers at the cycles with period 2 on some open neighborhood U_2 of f_0 in $\operatorname{Poly}_d^{\mathrm{mc}}(\mathbb{C})$.

First, the periodic points for f_0 with period 2 are the $\left(d^2-1\right)$th roots of unity that are not $(d-1)$th roots of unity. Choose representatives $w_0, \ldots, w_{\frac{d(d-1)}{2}-1}$ for the cycles for f_0 with period 2. Setting $\beta = \exp\left(\frac{2\pi i}{d^2-1}\right)$, we can take $w_j = \alpha^j \beta$ for all $j \in \{0, \ldots, d-2\}$ since these lie in pairwise distinct cycles for f_0 with period 2. We have $\left(f_0^{\circ 2}\right)'(w_j) = d^2$ for all $j \in \left\{0, \ldots, \frac{d(d-1)}{2}-1\right\}$. By the implicit function theorem, there exist an open neighborhood U_2 of f_0 in $\operatorname{Poly}_d^{\mathrm{mc}}(\mathbb{C})$ and holomorphic maps $\zeta_j^{(2)} \colon U_2 \to \mathbb{C}$, with $j \in \left\{0, \ldots, \frac{d(d-1)}{2}-1\right\}$, such that

$$\zeta_j^{(2)}(f_0) = w_j \quad \text{and} \quad \forall f \in U_2,\, f^{\circ 2}\left(\zeta_j^{(2)}(f)\right) = \zeta_j^{(2)}(f)$$

for all $j \in \left\{0, \ldots, \frac{d(d-1)}{2}-1\right\}$. Shrinking U_2 if necessary, we may assume that U_2 is connected and invariant under the action of $\mu_{d-1}(\mathbb{C}) = \langle \alpha \rangle$ by conjugation. For every $f \in U_2$, we have

$$\Phi_f^{(2)}(z) = \prod_{j=0}^{\frac{d(d-1)}{2}-1} \left(z - \zeta_j^{(2)}(f)\right)\left(z - f\left(\zeta_j^{(2)}(f)\right)\right) .$$

Now, for $j \in \left\{0, \ldots, \frac{d(d-1)}{2}-1\right\}$, define the holomorphic map $\rho_j^{(2)} \colon U_2 \to \mathbb{C}$ by

$$\rho_j^{(2)}(f) = \left(f^{\circ 2}\right)'\left(\zeta_j^{(2)}(f)\right) .$$

For every $f \in U_2$, we have

$$\Lambda_f^{(2)} = \left[\rho_0^{(2)}(f), \ldots, \rho_{\frac{d(d-1)}{2}-1}^{(2)}(f)\right] .$$

Define the holomorphic map

$$\boldsymbol{\rho}_2 = \left(\rho_0^{(2)}, \ldots, \rho_{\frac{d(d-1)}{2}-1}^{(2)}\right) \colon U_2 \to \mathbb{C}^{\frac{d(d-1)}{2}} .$$

Also denote by $\cdot$ the natural action of $\mathfrak{S}_{\frac{d(d-1)}{2}}$ on $\mathbb{C}^{\frac{d(d-1)}{2}}$, which is given by

$$\tau \cdot \left(\lambda_0, \ldots, \lambda_{\frac{d(d-1)}{2}-1}\right) = \left(\lambda_{\tau^{-1}(0)}, \ldots, \lambda_{\tau^{-1}\left(\frac{d(d-1)}{2}-1\right)}\right) .$$

By the formula above,

$$\forall f, g \in U_2,\, \Lambda_f^{(2)} = \Lambda_g^{(2)} \iff \left(\exists \tau \in \mathfrak{S}_{\frac{d(d-1)}{2}},\, \boldsymbol{\rho}_2(g) = \tau \cdot \boldsymbol{\rho}_2(f)\right) . \tag{5.3}$$

We may describe the behavior of $\boldsymbol{\rho}_2$ under conjugation. Define $\tau_0 \in \mathfrak{S}_{\frac{d(d-1)}{2}}$ to be the permutation such that, for each $j \in \left\{0, \ldots, \frac{d(d-1)}{2}-1\right\}$, the point αw_j lies in the cycle for f_0 containing $w_{\tau_0(j)}$. By the choice of $w_0, \ldots, w_{d-2}$, we have

$$\forall j \in \{0, \ldots, d-2\},\, \tau_0(j) = j+1 \pmod{d-1} \in \{0, \ldots, d-2\} . \tag{5.4}$$

Claim 76. We have $\boldsymbol{\rho}_2\left(\alpha^k \centerdot f\right) = \tau_0^k \cdot \boldsymbol{\rho}_2(f)$ for all $f \in U_2$ and all $k \in \{0, \ldots, d-2\}$.

Proof. For every $j \in \left\{0, \ldots, \frac{d(d-1)}{2} - 1\right\}$ and every $k \in \{0, \ldots, d-2\}$, there exists $\ell \in \left\{0, \ldots, \frac{d(d-1)}{2} - 1\right\}$ such that $\alpha^{-k}\zeta_j^{(2)}\left(\alpha^k \centerdot f\right)$ lies in the cycle for f containing $\zeta_\ell^{(2)}(f)$ for all $f \in U_2$, and we have $\ell = \tau_0^{-k}(j)$ since $\alpha^{-k} w_j$ lies in the cycle for f_0 containing w_ℓ by taking $f = f_0$. Thus, for each $f \in U_2$ and each $k \in \{0, \ldots, d-2\}$, the point $\alpha^{-k}\zeta_j^{(2)}\left(\alpha^k \centerdot f\right)$ belongs to the cycle for f containing $\zeta_{\tau_0^{-k}(j)}^{(2)}(f)$ for each $j \in \left\{0, \ldots, \frac{d(d-1)}{2} - 1\right\}$, and hence $\boldsymbol{\rho}_2\left(\alpha^k \centerdot f\right) = \tau_0^k \cdot \boldsymbol{\rho}_2(f)$ because the multiplier is invariant under conjugation. This completes the proof of the claim. □

Finally, define

$$A_2 = \left(\frac{\partial \rho_j^{(2)}}{\partial a_k}(f_0)\right)_{\substack{0 \leq j \leq \frac{d(d-1)}{2} - 1 \\ 0 \leq k \leq d-2}} \in M_{\frac{d(d-1)}{2} \times (d-1)}(\mathbb{C})$$

to be the Jacobian matrix of $\boldsymbol{\rho}_2$ at f_0. By Lemma 73, we have

$$A_2 = \left(d(k-d)\left(w_j^{k-d} + w_j^{d(k-d)}\right)\right)_{\substack{0 \leq j \leq \frac{d(d-1)}{2} - 1 \\ 0 \leq k \leq d-2}} . \tag{5.5}$$

5.4. **Proof of Theorem C.** Finally, let us conclude here our proof of Lemma 71 and deduce Theorem C.

By (5.2) and Claim 75, to prove Lemma 71, it suffices to show that there exists a nonempty open subset $U \subseteq U_1 \setminus \boldsymbol{\rho}_1^{-1}(\Delta)$ of $\mathrm{Poly}_d^{\mathrm{mc}}(\mathbb{C})$ such that

$$\forall f \in U,\ \left\{\sigma \in \mathfrak{S}_{d-1} : \Lambda_{\sigma * f}^{(2)} = \Lambda_f^{(2)}\right\} = \langle \sigma_0 \rangle . \tag{5.6}$$

Choose a connected open neighborhood $U_0 \subseteq U_1 \cap U_2$ of f_0 in $\mathrm{Poly}_d^{\mathrm{mc}}(\mathbb{C})$ such that $V_0 = \boldsymbol{\rho}_1(U_0) \subseteq \mathbb{C}^{d-1}$ is invariant under the natural action of $\mathfrak{S}_{d-1}$, and define the holomorphic map

$$\boldsymbol{\rho} = \boldsymbol{\rho}_2 \circ \boldsymbol{\rho}_1^{-1} \colon V_0 \to \mathbb{C}^{\frac{d(d-1)}{2}} .$$

By (5.3), for every $f \in U_0$ and every $\sigma \in \mathfrak{S}_{d-1}$,

$$\begin{aligned} \Lambda_{\sigma * f}^{(2)} = \Lambda_f^{(2)} &\iff \exists \tau \in \mathfrak{S}_{\frac{d(d-1)}{2}},\ \boldsymbol{\rho}_2(\sigma * f) = \tau \cdot \boldsymbol{\rho}_2(f) \\ &\iff \exists \tau \in \mathfrak{S}_{\frac{d(d-1)}{2}},\ \boldsymbol{\rho}\left(\sigma \cdot \boldsymbol{\rho}_1(f)\right) = \tau \cdot \boldsymbol{\rho}\left(\boldsymbol{\rho}_1(f)\right) . \end{aligned}$$

Thus, to prove Lemma 71, it suffices to show that there is a nonempty open subset $V \subseteq V_0$ of $\mathbb{C}^{d-1}$ such that

$$\forall \boldsymbol{\lambda} \in V,\ \left\{\sigma \in \mathfrak{S}_{d-1} : \exists \tau \in \mathfrak{S}_{\frac{d(d-1)}{2}},\ \boldsymbol{\rho}(\sigma \cdot \boldsymbol{\lambda}) = \tau \cdot \boldsymbol{\rho}(\boldsymbol{\lambda})\right\} = \langle \sigma_0 \rangle , \tag{5.7}$$

since $U = \boldsymbol{\rho}_1^{-1}(V \setminus \Delta)$ would then satisfy (5.6).

By Claims 75 and 76, for every $f \in U_0$ and every $k \in \{0, \ldots, d-2\}$, we have

$$\boldsymbol{\rho}\left(\sigma_0^k \cdot \boldsymbol{\rho}_1(f)\right) = \boldsymbol{\rho}_2\left(\sigma_0^k * f\right) = \boldsymbol{\rho}_2\left(\alpha^k \centerdot f\right) = \tau_0^k \cdot \boldsymbol{\rho}_2(f) = \tau_0^k \cdot \boldsymbol{\rho}\left(\boldsymbol{\rho}_1(f)\right) .$$

Therefore, we have

$$\forall \boldsymbol{\lambda} \in V_0,\ \forall k \in \{0, \ldots, d-2\},\ \boldsymbol{\rho}\left(\sigma_0^k \cdot \boldsymbol{\lambda}\right) = \tau_0^k \cdot \boldsymbol{\rho}(\boldsymbol{\lambda}) . \tag{5.8}$$

Thus, in view of (5.7), it is enough to prove that there is a nonempty open subset $V \subseteq V_0$ of $\mathbb{C}^{d-1}$ such that $\boldsymbol{\rho}(\sigma \cdot \boldsymbol{\lambda}) \neq \tau \cdot \boldsymbol{\rho}(\boldsymbol{\lambda})$ for all $\boldsymbol{\lambda} \in V$, all $\sigma \in \mathfrak{S}_{d-1} \setminus \langle \sigma_0 \rangle$ and all $\tau \in \mathfrak{S}_{\frac{d(d-1)}{2}}$.

Now, identify elements of $\mathbb{C}^{d-1}$ and $\mathbb{C}^{\frac{d(d-1)}{2}}$ with column vectors. Define

$$A = A_2 A_1^{-1} \in M_{\frac{d(d-1)}{2} \times (d-1)}(\mathbb{C})$$

to be the Jacobian matrix of $\boldsymbol{\rho}$ at $(d, \ldots, d) \in \mathbb{C}^{d-1}$, so that

$$\boldsymbol{\rho}\left((d, \ldots, d) + h\right) = \left(d^2, \ldots, d^2\right) + Ah + o(h) \quad \text{as} \quad h \to 0\,.$$

Denote by $\times_{\mathrm{c}}$ the action of $\mathfrak{S}_{d-1}$ on $M_{\frac{d(d-1)}{2} \times (d-1)}(\mathbb{C})$ that permutes the columns of matrices, which is given by

$$\sigma \times_{\mathrm{c}} \left(C_0 \;\middle|\; \ldots \;\middle|\; C_{d-2}\right) = \left(C_{\sigma^{-1}(0)} \;\middle|\; \ldots \;\middle|\; C_{\sigma^{-1}(d-2)}\right)\,.$$

Then $M(\sigma \cdot h) = \left(\sigma^{-1} \times_{\mathrm{c}} M\right) h$ for all $\sigma \in \mathfrak{S}_{d-1}$, all $M \in M_{\frac{d(d-1)}{2} \times (d-1)}(\mathbb{C})$ and all $h \in \mathbb{C}^{d-1}$. Therefore, for every $\sigma \in \mathfrak{S}_{d-1}$, we have

$$\boldsymbol{\rho}\left((d, \ldots, d) + \sigma \cdot h\right) = \left(d^2, \ldots, d^2\right) + \left(\sigma^{-1} \times_{\mathrm{c}} A\right) h + o(h) \quad \text{as} \quad h \to 0\,,$$

and hence the Jacobian matrix of $\boldsymbol{\lambda} \mapsto \boldsymbol{\rho}(\sigma \cdot \boldsymbol{\lambda})$ at $(d, \ldots, d)$ equals $\sigma^{-1} \times_{\mathrm{c}} A$. Also denote by $\times_{\mathrm{r}}$ the action of $\mathfrak{S}_{\frac{d(d-1)}{2}}$ on $M_{\frac{d(d-1)}{2} \times (d-1)}(\mathbb{C})$ that permutes the rows of matrices, which is given by

$$\tau \times_{\mathrm{r}} \begin{pmatrix} R_0 \\ \hline \vdots \\ \hline R_{\frac{d(d-1)}{2}-1} \end{pmatrix} = \begin{pmatrix} R_{\tau^{-1}(0)} \\ \hline \vdots \\ \hline R_{\tau^{-1}\left(\frac{d(d-1)}{2}-1\right)} \end{pmatrix}\,.$$

Then $\tau \cdot (Mh) = (\tau \times_{\mathrm{r}} M)\, h$ for all $\tau \in \mathfrak{S}_{\frac{d(d-1)}{2}}$, all $M \in M_{\frac{d(d-1)}{2} \times (d-1)}(\mathbb{C})$ and all $h \in \mathbb{C}^{d-1}$. Therefore, for every $\tau \in \mathfrak{S}_{\frac{d(d-1)}{2}}$, we have

$$\tau \cdot \boldsymbol{\rho}\left((d, \ldots, d) + h\right) = \left(d^2, \ldots, d^2\right) + (\tau \times_{\mathrm{r}} A)\, h + o(h) \quad \text{as} \quad h \to 0\,,$$

and hence the Jacobian matrix of $\boldsymbol{\lambda} \mapsto \tau \cdot \boldsymbol{\rho}(\boldsymbol{\lambda})$ at $(d, \ldots, d)$ equals $\tau \times_{\mathrm{r}} A$.

Thus, Lemma 71 follows easily from the result below.

Lemma 77. *We have*

$$\left\{\sigma \in \mathfrak{S}_{d-1} : \exists \tau \in \mathfrak{S}_{\frac{d(d-1)}{2}},\, \sigma \times_c A = \tau \times_r A\right\} = \langle \sigma_0 \rangle\,.$$

Let us postpone the proof of Lemma 77 and finish our proof of Lemma 71 first.

Proof of Lemma 71. For each $\sigma \in \mathfrak{S}_{d-1} \setminus \langle \sigma_0 \rangle$ and each $\tau \in \mathfrak{S}_{\frac{d(d-1)}{2}}$, the Jacobian matrices of $\boldsymbol{\lambda} \mapsto \boldsymbol{\rho}(\sigma \cdot \boldsymbol{\lambda})$ and $\boldsymbol{\lambda} \mapsto \tau \cdot \boldsymbol{\rho}(\boldsymbol{\lambda})$ at $(d, \ldots, d)$ equal $\sigma^{-1} \times_{\mathrm{c}} A$ and $\tau \times_{\mathrm{r}} A$, respectively, and these two matrices are different by Lemma 77. In particular, for every $\sigma \in \mathfrak{S}_{d-1} \setminus \langle \sigma_0 \rangle$ and every $\tau \in \mathfrak{S}_{\frac{d(d-1)}{2}}$, the holomorphic maps $\boldsymbol{\lambda} \mapsto \boldsymbol{\rho}(\sigma \cdot \boldsymbol{\lambda})$ and $\boldsymbol{\lambda} \mapsto \tau \cdot \boldsymbol{\rho}(\boldsymbol{\lambda})$ are different, and therefore $\{\boldsymbol{\lambda} \in V_0 : \boldsymbol{\rho}(\sigma \cdot \boldsymbol{\lambda}) \neq \tau \cdot \boldsymbol{\rho}(\boldsymbol{\lambda})\}$ forms a dense open subset of V_0. As a result,

$$V = \bigcap_{\sigma \in \mathfrak{S}_{d-1} \setminus \langle \sigma_0 \rangle} \; \bigcap_{\tau \in \mathfrak{S}_{\frac{d(d-1)}{2}}} \{\boldsymbol{\lambda} \in V_0 : \boldsymbol{\rho}(\sigma \cdot \boldsymbol{\lambda}) \neq \tau \cdot \boldsymbol{\rho}(\boldsymbol{\lambda})\}$$

is a nonempty open subset of $\mathbb{C}^{d-1}$, which is contained in V_0. Set $U = \boldsymbol{\rho}_1^{-1}(V \setminus \Delta)$. Then U is a nonempty open subset of $\operatorname{Poly}_d^{\mathrm{mc}}(\mathbb{C})$ and we have

$$\left\{g \in \operatorname{Poly}_d^{\mathrm{mc}}(\mathbb{C}) : \Lambda_g^{(1)} = \Lambda_f^{(1)} \text{ and } \Lambda_g^{(2)} = \Lambda_f^{(2)}\right\} = \left\{\alpha^k \centerdot f : k \in \{0, \dots, d-2\}\right\}$$

for all $f \in U$ by the previous discussion. Thus, the lemma is proved. □

To complete our proof of Lemma 71, it remains to prove Lemma 77. To do this, we shall first show the result below. From now on, write

$$A = \left(A_{j,k}\right)_{\substack{0 \leq j \leq \frac{d(d-1)}{2}-1 \\ 0 \leq k \leq d-2}} .$$

Claim 78. The entries $A_{j,0}$, with $j \in \left\{0, \dots, \frac{d(d-1)}{2} - 1\right\}$, of the first column of A are pairwise distinct.

Proof. By Claim 74 and (5.5), we have

$$A_{j,k} = d\left(\frac{P_k\left(w_j\right)}{w_j^d} + \frac{P_k\left(w_j^d\right)}{w_j^{d^2}}\right)$$

for all $j \in \left\{0, \dots, \frac{d(d-1)}{2} - 1\right\}$ and all $k \in \{0, \dots, d-2\}$. In particular, we have

$$A_{j,0} = d\left(\frac{P_0\left(w_j\right)}{w_j^d} + \frac{P_0\left(w_j^d\right)}{w_j^{d^2}}\right) = \left(\frac{d}{d-1}\right) F\left(w_j\right)$$

for all $j \in \left\{0, \dots, \frac{d(d-1)}{2} - 1\right\}$, where

$$F(T) = (d-1)\left(\frac{P_0(T)}{T^d} + \frac{P_0\left(T^d\right)}{T^{d^2}}\right) \in \mathbb{C}(T) .$$

Recall that the points w_j, with $j \in \left\{0, \dots, \frac{d(d-1)}{2} - 1\right\}$, are representatives for the cycles for f_0 with period 2. Moreover, the periodic points for f_0 with period 2 are given by β^j, with $j \in \mathbb{Z} \setminus (d+1)\mathbb{Z}$, where $\beta = \exp\left(\frac{2\pi i}{d^2-1}\right)$ as before. Therefore, it suffices to show that

$$\forall j, k \in \mathbb{Z} \setminus (d+1)\mathbb{Z},\ F\left(\beta^j\right) = F\left(\beta^k\right) \Longrightarrow k \equiv j \text{ or } jd \pmod{d^2-1} .$$

Now, note that $P_0(T) = \frac{1}{d-1}\left(\frac{T^{d-1}-1}{T-1}\right)$, which yields

$$F(T) = \frac{T^{d-1}-1}{T^d(T-1)} + \frac{T^{d(d-1)}-1}{T^{d^2}\left(T^d-1\right)} .$$

Therefore, for every $j \in \mathbb{Z} \setminus (d+1)\mathbb{Z}$, we have

$$F\left(\beta^j\right) = \frac{\beta^{j(d-1)}-1}{\beta^{jd}\left(\beta^j-1\right)} + \frac{1-\beta^{j(d-1)}}{\beta^{jd}\left(\beta^{jd}-1\right)} = \frac{\left(\beta^{j(d-1)}-1\right)^2}{\beta^{j(d-1)}\left(\beta^j-1\right)\left(\beta^{jd}-1\right)}$$

since $\beta^{d^2} = \beta$, and hence

$$F\left(\beta^j\right) = \frac{\left(\beta^{\frac{j(d-1)}{2}} - \beta^{\frac{-j(d-1)}{2}}\right)^2}{\beta^{\frac{j(d+1)}{2}}\left(\beta^{\frac{j}{2}} - \beta^{\frac{-j}{2}}\right)\left(\beta^{\frac{jd}{2}} - \beta^{\frac{-jd}{2}}\right)} = \frac{\sin\left(\frac{j\pi}{d+1}\right)^2 \exp\left(\frac{-j\pi i}{d-1}\right)}{\sin\left(\frac{j\pi}{d^2-1}\right)\sin\left(\frac{jd\pi}{d^2-1}\right)} .$$

Now, suppose that $j, k \in \mathbb{Z} \setminus (d+1)\mathbb{Z}$ are such that $F\left(\beta^j\right) = F\left(\beta^k\right)$. Then there exists $\ell \in \mathbb{Z}$ such that

$$\frac{-j\pi}{d-1} = \frac{-k\pi}{d-1} + \ell\pi \quad \text{and} \quad \frac{\sin\left(\frac{j\pi}{d+1}\right)^2}{\sin\left(\frac{j\pi}{d^2-1}\right)\sin\left(\frac{jd\pi}{d^2-1}\right)} = \frac{(-1)^\ell \sin\left(\frac{k\pi}{d+1}\right)^2}{\sin\left(\frac{k\pi}{d^2-1}\right)\sin\left(\frac{kd\pi}{d^2-1}\right)}\,.$$

Using basic trigonometric identities, it follows that

$$k = j + \ell(d-1) \quad \text{and} \quad \frac{1-\cos\left(\frac{j\pi}{d+1}\right)^2}{\cos\left(\frac{j\pi}{d+1}\right) - \cos\left(\frac{j\pi}{d-1}\right)} = \frac{1-\cos\left(\frac{(j-2\ell)\pi}{d+1}\right)^2}{\cos\left(\frac{(j-2\ell)\pi}{d+1}\right) - \cos\left(\frac{j\pi}{d-1}\right)}\,.$$

Now, setting $a = \cos\left(\frac{j\pi}{d-1}\right)$, note that the function $\varphi\colon (-1,1) \setminus \{a\} \to \mathbb{R}$ given by $\varphi(x) = \frac{1-x^2}{x-a}$ is injective because $a \in [-1,1]$. Therefore, $\cos\left(\frac{j\pi}{d+1}\right) = \cos\left(\frac{(j-2\ell)\pi}{d+1}\right)$, which yields $\frac{(j-2\ell)\pi}{d+1} \equiv \pm\frac{j\pi}{d+1} \pmod{2\pi}$, and hence $\ell \equiv 0$ or $j \pmod{d+1}$. Thus, $k \equiv j$ or $jd \pmod{d^2-1}$, and the claim is proved. □

We shall now prove Lemma 77.

Proof of Lemma 77. For each $k \in \{0, \ldots, d-2\}$, the two maps $\boldsymbol{\lambda} \mapsto \boldsymbol{\rho}\left(\sigma_0^k \cdot \boldsymbol{\lambda}\right)$ and $\boldsymbol{\lambda} \mapsto \tau_0^k \cdot \boldsymbol{\rho}(\boldsymbol{\lambda})$ coincide according to (5.8), and hence they have the same Jacobian matrix at $(d, \ldots, d)$. Thus, by the previous discussion,

$$\forall k \in \{0, \ldots, d-2\},\ \sigma_0^{-k} \times_{\mathrm{c}} A = \tau_0^k \times_{\mathrm{r}} A\,. \tag{5.9}$$

Now, suppose that $\sigma \in \mathfrak{S}_{d-1}$ and $\tau \in \mathfrak{S}_{\frac{d(d-1)}{2}}$ satisfy $\sigma \times_{\mathrm{c}} A = \tau \times_{\mathrm{r}} A$. Write

$$\sigma \times_{\mathrm{c}} A = (M_{j,k})_{\substack{0 \leq j \leq \frac{d(d-1)}{2}-1 \\ 0 \leq k \leq d-2}} = \tau \times_{\mathrm{r}} A\,.$$

Set $\ell = \sigma(0)$, and let us prove that $\sigma = \sigma_0^\ell$. By (5.9), we have

$$A_{0,0} = M_{0,\ell} = A_{\tau^{-1}(0),\ell} = A_{\tau^{-1}(0),\sigma_0^\ell(0)} = A_{\tau_0^{-\ell}\tau^{-1}(0),0}\,,$$

which yields $\tau_0^{-\ell}\tau^{-1}(0) = 0$ by Claim 78, and hence $\tau^{-1}(0) = \ell$ according to (5.4). As a result, for every $k \in \{0, \ldots, d-2\}$, we have

$$A_{0,\sigma^{-1}(k)} = M_{0,k} = A_{\ell,k} = A_{\tau_0^\ell(0),k} = A_{0,\sigma_0^{-\ell}(k)}$$

by (5.4) and (5.9). Moreover, $A_{0,k} = A_{\tau_0^{-k}(0),0}$ for all $k \in \{0, \ldots, d-2\}$ by (5.9), and hence $A_{0,0}, \ldots, A_{0,d-2}$ are pairwise distinct by (5.4) and Claim 78. Therefore, $\sigma^{-1}(k) = \sigma_0^{-\ell}(k)$ for all $k \in \{0, \ldots, d-2\}$, and hence $\sigma = \sigma_0^\ell$. Thus, the lemma is proved. □

Thus, our proof of Lemma 71 is complete. To conclude, we simply observe that Theorem C follows immediately. For completeness, we include details.

Proof of Theorem C. For $P \geq 1$, denote again by $\Sigma_d^{(P)}$ the scheme-theoretic image of $\mathrm{Mult}_d^{(P)}$. Then we have

$$\dim\left(\Sigma_d^{(2)}\right) \geq \dim\left(\Sigma_d^{(1)}\right) = d-1 = \dim\left(\mathcal{P}_d\right)$$

because $\Sigma_d^{(1)}$ is the image of $\Sigma_d^{(2)}$ under the projection $p_1\colon \mathbb{A}^d \times \mathbb{A}^{\frac{d(d-1)}{2}} \to \mathbb{A}^d$ onto the first factor. Consequently, the induced morphism $\mathrm{Mult}_d^{(2)}\colon \mathcal{P}_d \to \Sigma_d^{(2)}$ has some finite degree $D \geq 1$. There is a nonempty Zariski-open subset W of $\Sigma_d^{(2)}$ such that every $\boldsymbol{\sigma} \in W(\mathbb{C})$ has exactly D preimages under $\mathrm{Mult}_d^{(2)} \colon \mathcal{P}_d(\mathbb{C}) \to \Sigma_d^{(2)}(\mathbb{C})$. Now, consider the preimage V of W under $\mathrm{Mult}_d^{(2)}$. Then V is a nonempty Zariski-open subset of $\mathcal{P}_d$. For every $[f] \in V(\mathbb{C})$, there are exactly D elements $[g] \in \mathcal{P}_d(\mathbb{C})$ such that $\Lambda_g^{(1)} = \Lambda_f^{(1)}$ and $\Lambda_g^{(2)} = \Lambda_f^{(2)}$. By Lemma 71, there is a nonempty open subset U of $\mathrm{Poly}_d^{\mathrm{mc}}(\mathbb{C})$ such that, for every $f \in U$, the elements $g \in \mathrm{Poly}_d^{\mathrm{mc}}(\mathbb{C})$ such that $\Lambda_g^{(1)} = \Lambda_f^{(1)}$ and $\Lambda_g^{(2)} = \Lambda_f^{(2)}$ are precisely the $\alpha^k \centerdot f$, with $k \in \{0, \dots, d-2\}$. Now, consider the image V' of U under the holomorphic map $\pi_d^{\mathrm{mc}} \colon \mathrm{Poly}_d^{\mathrm{mc}}(\mathbb{C}) \to \mathcal{P}_d(\mathbb{C})$, via the biholomorphism $\mathcal{P}_d(\mathbb{C}) \cong \mathcal{P}_d^{\mathrm{mc}}(\mathbb{C})$. Then V' is a nonempty open subset of $\mathcal{P}_d(\mathbb{C})$. Furthermore, each $[f] \in V'$ is the unique $[g] \in \mathcal{P}_d(\mathbb{C})$ such that $\Lambda_g^{(1)} = \Lambda_f^{(1)}$ and $\Lambda_g^{(2)} = \Lambda_f^{(2)}$. Finally, note that $V(\mathbb{C}) \cap V' \neq \varnothing$, as $V(\mathbb{C})$ is dense in $\mathcal{P}_d(\mathbb{C})$ for the complex topology. Thus, we have $D = 1$, and the theorem is proved. ☐

Appendix A. Additional estimates on multipliers of polynomial maps

In this section, we again fix an integer $d \geq 2$.

A.1. Upper bounds on absolute values of multipliers.

As mentioned in the introduction, it is not difficult to obtain upper bounds on the characteristic exponents of polynomial maps at periodic points in terms of the maximal escape rate. For completeness, let us give here some details.

Given a valued field K, we denote by $|.|_K$ its absolute value and by $\|.\|_{K^{d-1}}$ the norm on K^{d-1} given by

$$\|\boldsymbol{c}\|_{K^{d-1}} = \max_{j \in \{1, \dots, d-1\}} |c_j|_K \quad \text{for} \quad \boldsymbol{c} = (c_1, \dots, c_{d-1}) \in K^{d-1}\,.$$

We shall also work again with the normal form introduced by Ingram in [Ing12]. Given a field K of characteristic 0 and $\boldsymbol{c} = (c_1, \dots, c_{d-1}) \in K^{d-1}$, we define

$$f_{\boldsymbol{c}}(z) = \frac{1}{d} z^d + \sum_{j=1}^{d-1} \frac{(-1)^j \tau_j(\boldsymbol{c})}{d-j} z^{d-j} \in \mathrm{Poly}_d(K)\,,$$

where $\tau_1(\boldsymbol{c}), \dots, \tau_{d-1}(\boldsymbol{c})$ are the elementary symmetric functions of $c_1, \dots, c_{d-1}$.

For our purposes, we shall use the generalization of Claims 51 and 53 below. As it can be derived in a similar way, by using the triangle inequality and elimination theory, we omit the proof.

Claim 79. Assume that K is an algebraically closed valued field of characteristic 0. Then there exist $\delta_K \in \mathbb{R}_{>0}$ and $\Delta_K \in \mathbb{R}$ both depending only on the restriction of $|.|_K$ to $\mathbb{Q}$ such that $g_{f_{\boldsymbol{c}}}(z) \geq \log^+|z|_K + \Delta_K$ for all $\boldsymbol{c} \in K^{d-1}$ and all $z \in K$ such that $|z|_K > \delta_K \|\boldsymbol{c}\|_{K^{d-1}}$. In addition, there exists some $E_K \in \mathbb{R}$ depending only on the restriction of $|.|_K$ to $\mathbb{Q}$ such that $M_{f_{\boldsymbol{c}}} \geq \log^+\|\boldsymbol{c}\|_{K^{d-1}} + E_K$ for all $\boldsymbol{c} \in K^{d-1}$. Furthermore, we can take $\delta_K = 1$, $\Delta_K = 0$ and $E_K = 0$ if K is non-Archimedean with residue characteristic 0 or greater than d.

Finally, we obtain the following result:

Proposition 80. *Assume that K is an algebraically closed valued field of characteristic 0. Then there exists $B_K \in \mathbb{R}$ depending only on the restriction of $|.|_K$ to $\mathbb{Q}$ such that $M_f^{(p)} \leq (d-1)M_f + B_K$ for all $f \in \mathrm{Poly}_d(K)$ and all $p \geq 1$. Moreover,* *we can take $B_K = 0$ if K is non-Archimedean with residue characteristic either* 0 *or greater than d.*

Proof. By conjugation, we may restrict our attention to the polynomials $f_{\boldsymbol{c}}$, with $\boldsymbol{c} \in K^{d-1}$. Define

$$\varepsilon_K = \begin{cases} 1 + \max\left\{\delta_K, \exp\left(-\Delta_K\right)\right\} & \text{if } K \text{ is Archimedean} \\ \max\left\{1, \delta_K, \exp\left(-\Delta_K\right)\right\} & \text{if } K \text{ is non-Archimedean} \end{cases}$$

and

$$B_K = (d-1)\left(\log\left(\varepsilon_K\right) - E_K\right) \in \mathbb{R}\,,$$

with $\delta_K \in \mathbb{R}_{>0}$, $\Delta_K \in \mathbb{R}$ and $E_K \in \mathbb{R}$ as in Claim 79. Note that B_K depends only on the restriction of $|.|_K$ to $\mathbb{Q}$ and we have $B_K = 0$ if K is non-Archimedean with residue characteristic 0 or greater than d. Now, suppose that $\boldsymbol{c} \in K^{d-1}$, $p \geq 1$ and $z_0 \in K$ is a fixed point for $f_{\boldsymbol{c}}^{\circ p}$. For every $j \in \{0, \ldots, p-1\}$, as $g_{f_{\boldsymbol{c}}}\left(f_{\boldsymbol{c}}^{\circ j}\left(z_0\right)\right) = 0$, we have

$$\begin{aligned} \left|f_{\boldsymbol{c}}^{\circ j}\left(z_0\right)\right|_K &\leq \max\left\{\delta_K \|\boldsymbol{c}\|_{K^{d-1}}, \exp\left(-\Delta_K\right)\right\} \\ &\leq \max\left\{\delta_K, \exp\left(-\Delta_K\right)\right\} \cdot \max\left\{1, \|\boldsymbol{c}\|_{K^{d-1}}\right\} \end{aligned}$$

by Claim 79. Therefore, for every $j \in \{0, \ldots, p-1\}$, we have

$$\left|f_{\boldsymbol{c}}'\left(f_{\boldsymbol{c}}^{\circ j}\left(z_0\right)\right)\right|_K = \prod_{\ell=1}^{d-1} \left|f_{\boldsymbol{c}}^{\circ j}\left(z_0\right) - c_\ell\right|_K \leq \varepsilon_K^{d-1} \max\left\{1, \|\boldsymbol{c}\|_{K^{d-1}}\right\}^{d-1}$$

by the triangle inequality. As a result, we have

$$\begin{aligned} \frac{1}{p}\log\left|\left(f_{\boldsymbol{c}}^{\circ p}\right)'\left(z_0\right)\right|_K &= \frac{1}{p}\sum_{j=0}^{p-1} \log\left|f_{\boldsymbol{c}}'\left(f_{\boldsymbol{c}}^{\circ j}\left(z_0\right)\right)\right|_K \\ &\leq (d-1)\log^+\|\boldsymbol{c}\|_{K^{d-1}} + (d-1)\log\left(\varepsilon_K\right) \\ &\leq (d-1)M_{f_{\boldsymbol{c}}} + B_K \end{aligned}$$

by Claim 79. This completes the proof of the proposition. □

Remark 81. In [Buf03], Buff used de Branges's theorem to show that we can take $B_{\mathbb{C}} = 2\log(d)$ in the statement of Proposition 80.

As an immediate consequence of Proposition 80, we obtain an upper bound on the heights of multipliers of polynomial maps in terms of the critical height.

Corollary 82. *There exists some $B \in \mathbb{R}$ such that $H_f^{(p)} \leq (d-1)H_f + B$ for all $f \in \operatorname{Poly}_d\left(\overline{\mathbb{Q}}\right)$ and all $p \geq 1$.*

A.2. **Lower bounds on absolute values of multipliers.** Finally, let us obtain here a lower bound on the absolute values of multipliers of polynomial maps whose critical points all escape.

Suppose that $f \in \operatorname{Poly}_d(K)$, with K an algebraically closed valued field of characteristic 0. We define the *minimal escape rate* m_f of f by

$$m_f = \min\left\{g_f(c) : c \in K,\ f'(c) = 0\right\}\,.$$

Also, for $p \geq 1$, we define

$$m_f^{(p)} = \min_{\lambda \in \Lambda_f^{(p)}} \left(\frac{1}{p}\log^+|\lambda|\right)\,,$$

where $|.|$ denotes the absolute value on K.

As a consequence of Lemma 32 in the Archimedean case and Lemma 61 in the non-Archimedean case, we obtain the result below. In the complex setting, it is a slightly weaker version of [EL92, Theorem 1.6].

Proposition 83. *Assume that K is an algebraically closed valued field of characteristic 0 that is either Archimedean or non-Archimedean with residue characteristic 0 or greater than d and $f \in \operatorname{Poly}_d(K)$. Then $m_f^{(p)} \geq (d-1)m_f$ for all $p \geq 1$.*

Proof. First, assume that K is endowed with an Archimedean absolute value $|.|_\infty$. Then, by Ostrowski's theorem, there exist an embedding $\sigma\colon K \hookrightarrow \mathbb{C}$ and $s \in (0,1]$ such that $|z|_\infty = |\sigma(z)|^s$ for each $z \in K$, where $|.|$ is the usual absolute value on $\mathbb{C}$. We have $m_f = s \cdot m_{\sigma(f)}$ and $m_f^{(p)} = s \cdot m_{\sigma(f)}^{(p)}$ for each $p \geq 1$. Thus, replacing f by $\sigma(f)$ if necessary, we may assume that $f \in \operatorname{Poly}_d(\mathbb{C})$. Note that the desired result is immediate if $m_f = 0$. Now, suppose that $m_f > 0$. Choose an integer $k \geq 0$ such that $d^k m_f \geq M_f$. Suppose that $z_0 \in \mathbb{C}$ is a periodic point for f with period $p \geq 1$. For $j \in \{0, \dots, p-1\}$, denote by U_j and V_j the respective connected components of $\{g_f < m_f\}$ and $\{g_f < d \cdot m_f\}$ containing $f^{\circ j}(z_0)$. Then U_j contains no critical point for f for all $j \in \{0, \dots, p-1\}$. By the Riemann–Hurwitz formula, it follows that f induces a biholomorphism from U_j to $V_{j+1 \pmod p}$ for all $j \in \{0, \dots, p-1\}$. Therefore, by Lemma 32, we have

$$\frac{1}{p} \log \left|\left(f^{\circ p}\right)'(z_0)\right| \geq \frac{d-1}{p} \left(\sum_{j=0}^{p-1} \frac{1}{d_j}\right) d^k m_f \geq (d-1) m_f \,,$$

where $d_j \leq d^k$ is the degree of $f^{\circ k}\colon V_j \to \left\{g_f < d^{k+1} m_f\right\}$ for all $j \in \{0, \dots, p-1\}$. This completes the proof of the proposition in the Archimedean case.

Now, assume that K is endowed with a non-Archimedean absolute value $|.|$ and the associated residue characteristic either equals 0 or is greater than d. Note that the desired inequality is immediate if the absolute value $|.|$ is trivial or if $m_f = 0$. From now on, suppose that $|.|$ is not trivial and $m_f > 0$. Choose an integer $k \geq 0$ such that $d^k m_f \geq M_f$. Suppose that $z_0 \in K$ is a periodic point for f with period $p \geq 1$. We have $g_f(z_0) = 0$. By Lemma 54, the sets $\{g_f < m_f\}$ and $\{g_f < d \cdot m_f\}$ are finite unions of disks. Thus, for $j \in \{0, \dots, p-1\}$, define U_j and V_j to be the respective disk components of $\{g_f < m_f\}$ and $\{g_f < d \cdot m_f\}$ that contain $f^{\circ j}(z_0)$. Then U_j contains no critical point for f for each $j \in \{0, \dots, p-1\}$. As a result, it follows from Lemmas 47 and 49 that f induces a bijection from U_j to $V_{j+1 \pmod p}$ for each $j \in \{0, \dots, p-1\}$. Therefore, by Lemma 61, we have

$$\frac{1}{p} \log \left|\left(f^{\circ p}\right)'(z_0)\right| \geq \frac{d-1}{p} \left(\sum_{j=0}^{p-1} \frac{1}{d_j}\right) d^k m_f \geq (d-1) m_f \,,$$

where $d_j \leq d^k$ is the degree of $f^{\circ k}\colon V_j \to \left\{g_f < d^{k+1} m_f\right\}$ for all $j \in \{0, \dots, p-1\}$. This completes the proof of the proposition in the non-Archimedean case. □

Appendix B. About isospectral polynomial maps

B.1. **Examples of isospectral polynomial maps of composite degrees.** As mentioned in the introduction, the multiplier spectrum morphisms are not always isomorphisms onto their images. In fact, there are nonconjugate polynomial maps of any composite degree that have the same multiset of multipliers for each period. Although it is already known, let us detail this here for the reader's convenience.

To exhibit isospectral polynomial maps, one can use the following result:

Proposition 84 ([Pak19b, Lemma 2.1])**.** *Assume that K is an algebraically closed field of characteristic 0 and $f \in \mathrm{Poly}_d(K)$ and $g \in \mathrm{Poly}_e(K)$, with $d, e \geq 2$. Then $\Lambda_{f \circ g}^{(p)} = \Lambda_{g \circ f}^{(p)}$ for all $p \geq 1$.*

Proof. For each $p \geq 1$, we have $g \circ (f \circ g)^{\circ p} = (g \circ f)^{\circ p} \circ g$, and hence g sends any periodic point for $f \circ g$ in K with period p to a periodic point for $g \circ f$ in K with period dividing p. Similarly, f sends any periodic point for $g \circ f$ in K with period $p \geq 1$ to a periodic point for $f \circ g$ in K with period dividing p. Moreover, the map $f \circ g$ induces a permutation of its set of periodic points in K, which preserves the periods. Therefore, g induces an injection from the set of periodic points for $f \circ g$ in K into the set of periodic points for $g \circ f$ in K, which preserves the periods. In fact, this map induced by g is a bijection as $g \circ f$ also permutes its periodic points in K. Finally, for each periodic point $z_0 \in K$ for $f \circ g$ with period $p \geq 1$, we have

$$\begin{aligned} \left((g \circ f)^{\circ p}\right)'(g(z_0)) &= \left(\prod_{j=0}^{p-1} f'\left((g \circ f)^{\circ j} \circ g(z_0)\right)\right)\left(\prod_{j=0}^{p-1} g'\left(f \circ (g \circ f)^{\circ j} \circ g(z_0)\right)\right) \\ &= \left(\prod_{j=0}^{p-1} f'\left(g \circ (f \circ g)^{\circ j}(z_0)\right)\right)\left(\prod_{j=0}^{p-1} g'\left((f \circ g)^{\circ j}(z_0)\right)\right) \\ &= \left((f \circ g)^{\circ p}\right)'(z_0) \,. \end{aligned}$$

Thus, the map g induces a bijection from the set of periodic points for $f \circ g$ in K onto the set of periodic points for $g \circ f$ in K, which preserves the periods and the multipliers. This completes the proof of the proposition. □

As a direct consequence of Proposition 84, we obtain the following:

Corollary 85. *Suppose that $d \geq 2$ is not a prime number. Then, for each $P \geq 1$, the morphism $\mathrm{Mult}_d^{(P)}$ is not injective.*

Proof. Assume that $d = d_1 d_2$, with $d_1, d_2 \geq 2$, and define

$$f(z) = z^d + 1 \in \mathrm{Poly}_d(\mathbb{Q}) \quad \text{and} \quad g(z) = \left(z^{d_1} + 1\right)^{d_2} \in \mathrm{Poly}_d(\mathbb{Q}) \,.$$

Then we have $f = h_1 \circ h_2$ and $g = h_2 \circ h_1$, where $h_1(z) = z^{d_1} + 1 \in \mathrm{Poly}_{d_1}(\mathbb{Q})$ and $h_2(z) = z^{d_2} \in \mathrm{Poly}_{d_2}(\mathbb{Q})$, and therefore $\mathrm{Mult}_d^{(P)}([f]) = \mathrm{Mult}_d^{(P)}([g])$ for all $P \geq 1$ by Proposition 84. However, $[f] \neq [g]$ in $\mathcal{P}_d(\mathbb{Q})$ because f has only 1 critical point in $\mathbb{C}$ whereas g has exactly $d_1 + 1$ critical points in $\mathbb{C}$. This completes the proof of the corollary. □

Actually, it is suspected that Proposition 84 is the unique source of examples of nonconjugate isospectral polynomial maps (see [Pak19a, Problem 3.1]).

B.2. **The case of quartic polynomial maps.** Finally, as isospectral polynomial maps of degree 2 or 3 are necessarily conjugate, let us investigate the situation for polynomial maps of degree 4.

Using explicit expressions for the multiplier spectrum morphisms, we show here that the pairs of nonconjugate isospectral quartic polynomial maps all come from Proposition 84. More precisely, we obtain the following result:

Proposition 86. *Assume that K is an algebraically closed field of characteristic* 0 *and $f, g \in \mathrm{Poly}_4(K)$ satisfy $\Lambda_f^{(1)} = \Lambda_g^{(1)}$ and $\Lambda_f^{(2)} = \Lambda_g^{(2)}$. Then $[f] = [g]$ in $\mathcal{P}_4(K)$ or there exist $h_1, h_2 \in \mathrm{Poly}_2(K)$ such that $f = h_1 \circ h_2$ and $g = h_2 \circ h_1$.*

Proof. We shall first work with monic centered quartic polynomials. Recall that

$$\mathrm{Poly}_4^{\mathrm{mc}} = \left\{z^4 + a_2 z^2 + a_1 z + a_0\right\}$$

and that the algebraic group $\mu_3 = \left\{\omega : \omega^3 = 1\right\}$ acts on $\mathrm{Poly}_4^{\mathrm{mc}}$ by

$$\omega \centerdot \left(z^4 + a_2 z^2 + a_1 z + a_0\right) = z^4 + \omega^{-1} a_2 z^2 + a_1 z + \omega a_0 \,.$$

Therefore, as $\mathcal{P}_4^{\mathrm{mc}} = \mathrm{Poly}_4^{\mathrm{mc}} / \mu_3$, we have

$$\mathbb{Q}\left[\mathcal{P}_4^{\mathrm{mc}}\right] = \mathbb{Q}\left[\mathrm{Poly}_4^{\mathrm{mc}}\right]^{\mu_3} = \mathbb{Q}[\alpha, \beta, \gamma, \delta] \,,$$

where

$$\alpha = a_1 \,, \quad \beta = a_0^3 \,, \quad \gamma = a_2^3 \quad \text{and} \quad \delta = a_0 a_2 \,.$$

Now, for $h_1(z) = z^2 + c_1 \in \mathrm{Poly}_2^{\mathrm{mc}}$ and $h_2(z) = z^2 + c_2 \in \mathrm{Poly}_2^{\mathrm{mc}}$, we have

$$h_1 \circ h_2(z) = z^4 + a_2 z^2 + a_1 z + a_0 \in \mathrm{Poly}_4^{\mathrm{mc}} \,, \quad \text{with} \quad \begin{cases} a_0 = c_2^2 + c_1 \\ a_1 = 0 \\ a_2 = 2c_2 \end{cases} \,,$$

and $[h_1 \circ h_2] = [h_2 \circ h_1]$ in $\mathcal{P}_4^{\mathrm{mc}}$ if and only if $c_1^3 = c_2^3$. Thus, setting

$$\mathcal{S}_4^{\mathrm{mc}} = \left\{[h_1 \circ h_2] : h_1, h_2 \in \mathrm{Poly}_2^{\mathrm{mc}}\right\} \subseteq \mathcal{P}_4^{\mathrm{mc}}$$

and

$$\mathcal{L}_4^{\mathrm{mc}} = \left\{[h_1 \circ h_2] : h_1, h_2 \in \mathrm{Poly}_2^{\mathrm{mc}},\ [h_1 \circ h_2] = [h_2 \circ h_1]\right\} \subseteq \mathcal{S}_4^{\mathrm{mc}} \,,$$

we have

$$\mathcal{S}_4^{\mathrm{mc}} = \{\alpha = 0\} \quad \text{and} \quad \mathcal{L}_4^{\mathrm{mc}} = \{\alpha = 0\} \cap \left\{64\beta^3 - \gamma^2 + 12\gamma\delta - 48\delta^2 - 8\gamma = 0\right\} \,.$$

For simplicity, write $s_j = \sigma_{4,j}^{(1)}$ for $j \in \{1, \dots, 4\}$ and $t_j = \sigma_{4,j}^{(2)}$ for $j \in \{1, \dots, 6\}$, so that

$$\mathrm{Mult}_4^{(2)} = (s_1, \dots, s_4, t_1, \dots, t_6) : \mathcal{P}_4^{\mathrm{mc}} \to \mathbb{A}^{10}$$

via the natural isomorphism $\mathcal{P}_4 \cong \mathcal{P}_4^{\mathrm{mc}}$. Using the software SageMath, we obtain

$$\begin{aligned}
s_1 &= -8\alpha + 12 \,, \\
s_2 &= 18\alpha^2 - 60\alpha + 4\gamma - 16\delta + 48 \,, \\
s_4 &= -27\alpha^4 + 108\alpha^3 - 4\alpha^2\gamma + 144\alpha^2\delta - 144\alpha^2 + 8\alpha\gamma - 288\alpha\delta \\
&\quad + 16\gamma\delta - 128\delta^2 + 64\alpha + 256\beta + 128\delta \,, \\
t_2 &= 27\alpha^4 + 324\alpha^3 + 4\alpha^2\gamma - 144\alpha^2\delta + 1440\alpha^2 + 24\alpha\gamma - 864\alpha\delta \\
&\quad - 16\gamma\delta + 128\delta^2 + 2880\alpha - 256\beta + 96\gamma - 512\delta + 3840 \,.
\end{aligned}$$

Then, using elimination with the software SageMath, we obtain

- an expression for α as an element of $\mathbb{Q}\left[s_1\right]$:

$$\alpha = \frac{-1}{8} s_1 + \frac{3}{2} \,, \tag{B.1}$$

- a polynomial equation in δ with coefficients in $\mathbb{Q}[s_1, s_2, s_4, t_2]$, degree 1 and leading coefficient a constant multiple of $s_1 - 12$:

$$2048\,(s_1 - 12)\,\delta - 9s_1^3 + 660s_1^2 - 16s_1s_2 - 19952s_1 + 576s_2 - 16s_4 - 16t_2 + 202944 = 0\,, \tag{B.2}$$

- a polynomial equation in δ with coefficients in $\mathbb{Q}[s_1, s_2, s_4, t_2]$, degree 2 and constant leading coefficient:

$$\begin{aligned} 1048576\delta^2 &+ 512\left(315s_1^2 + 4s_2^2 - 3624s_1 - 352s_2 - 12s_4 + 4t_2 - 9552\right)\delta \\ &- 9s_1^2s_2^2 + 3672s_1^2s_2 + 81s_1^2s_4 - 63s_1^2t_2 - 24s_1s_2^2 - 344240s_1^2 - 124416s_1s_2 \\ &+ 1168s_1s_4 + 784s_1t_2 + 4848s_2^2 - 672s_2s_4 - 160s_2t_2 + s_4^2 + 2s_4t_2 + t_2^2 \\ &+ 7144320s_1 + 1537152s_2 - 12432s_4 - 11664t_2 - 12936960 = 0\,, \end{aligned} \tag{B.3}$$

- an expression for β as an element of $\mathbb{Q}[\delta, s_1, s_2, s_4, t_2]$:

$$\begin{aligned} \beta = &\frac{-3}{2048}\delta s_1^2 + \frac{3}{65536}s_1^2s_2 + \frac{1}{4}\delta^2 + \frac{31}{256}\delta s_1 - \frac{1}{64}\delta s_2 \\ &+ \frac{103}{16384}s_1^2 - \frac{1}{2048}s_1s_2 - \frac{1}{65536}s_1s_4 - \frac{1}{65536}s_1t_2 - \frac{127}{128}\delta \\ &- \frac{1007}{2048}s_1 + \frac{69}{4096}s_2 + \frac{55}{16384}s_4 - \frac{9}{16384}t_2 + \frac{7131}{1024}\,, \end{aligned} \tag{B.4}$$

- an expression for γ as an element of $\mathbb{Q}[\delta, s_1, s_2, s_4, t_2]$:

$$\gamma = \frac{-9}{128}s_1^2 + 4\delta - \frac{3}{16}s_1 + \frac{1}{4}s_2 + \frac{3}{8}\,. \tag{B.5}$$

In particular, by (B.1) and the discussion above, we have $\mathcal{S}_4^{\mathrm{mc}} = \{s_1 = 12\}$. Thus, by the relations (B.1), (B.2), (B.4) and (B.5), each element $[f] \in \mathcal{P}_4^{\mathrm{mc}}(K) \setminus \mathcal{S}_4^{\mathrm{mc}}(K)$ is the unique $[g] \in \mathcal{P}_4^{\mathrm{mc}}(K)$ such that $\mathrm{Mult}_4^{(2)}([g]) = \mathrm{Mult}_4^{(2)}([f])$. Now, note that every element of K^{10} has at most two preimages in $\mathcal{P}_4^{\mathrm{mc}}(K)$ under $\mathrm{Mult}_4^{(2)}$ by the relations (B.1), (B.3), (B.4) and (B.5). Moreover, for all $[f] = [h_1 \circ h_2] \in \mathcal{S}_4^{\mathrm{mc}}(K)$, with $h_1, h_2 \in \mathrm{Poly}_2^{\mathrm{mc}}(K)$, the elements $[h_1 \circ h_2]$ and $[h_2 \circ h_1]$ of $\mathcal{P}_4^{\mathrm{mc}}(K)$ are both preimages of $\mathrm{Mult}_4^{(2)}([f])$ under $\mathrm{Mult}_4^{(2)}$ by Proposition 84, and these elements are distinct if $[f] \in \mathcal{S}_4^{\mathrm{mc}}(K) \setminus \mathcal{L}_4^{\mathrm{mc}}(K)$. As a result, for every $[f] = [h_1 \circ h_2] \in \mathcal{S}_4^{\mathrm{mc}}(K)$, with $h_1, h_2 \in \mathrm{Poly}_2^{\mathrm{mc}}(K)$, we have

$$\forall [g] \in \mathcal{P}_4^{\mathrm{mc}}(K),\ \mathrm{Mult}_4^{(2)}([g]) = \mathrm{Mult}_4^{(2)}([f]) \Longleftrightarrow [g] = [h_1 \circ h_2] \ \text{ or } \ [h_2 \circ h_1]$$

since $\mathcal{L}_4^{\mathrm{mc}}$ is a proper Zariski-closed subset of $\mathcal{S}_4^{\mathrm{mc}}$.

Finally, assume that $f, g \in \mathrm{Poly}_4(K)$ satisfy $\Lambda_f^{(1)} = \Lambda_g^{(1)}$ and $\Lambda_f^{(2)} = \Lambda_g^{(2)}$. Then, using the natural isomorphism $\mathcal{P}_4 \cong \mathcal{P}_4^{\mathrm{mc}}$, it follows from the discussion above that $[f] = [g]$ in $\mathcal{P}_4(K)$ or there exist $h_1, h_2 \in \mathrm{Poly}_2^{\mathrm{mc}}(K)$ such that $[f] = [h_1 \circ h_2]$ and $[g] = [h_2 \circ h_1]$ in $\mathcal{P}_4(K)$. In the latter situation, there are $\phi, \psi \in \mathrm{Aff}(K)$ such that $f = \phi \centerdot (h_1 \circ h_2)$ and $g = \psi \centerdot (h_2 \circ h_1)$ in $\mathrm{Poly}_4(K)$, and this yields $f = k_1 \circ k_2$ and $g = k_2 \circ k_1$, where $k_1 = \phi \circ h_1 \circ \psi^{-1}$ and $k_2 = \psi \circ h_2 \circ \phi^{-1}$. Thus, the proposition is proved. □

References

[Bea91] Alan F. Beardon, *Iteration of rational functions*, Graduate Texts in Mathematics, vol. 132, Springer-Verlag, New York, 1991, Complex analytic dynamical systems. MR 1128089

[Ben03] Robert L. Benedetto, *Non-Archimedean holomorphic maps and the Ahlfors Islands theorem*, Amer. J. Math. **125** (2003), no. 3, 581–622. MR 1981035

[Ben08] ———, *An Ahlfors islands theorem for non-Archimedean meromorphic functions*, Trans. Amer. Math. Soc. **360** (2008), no. 8, 4099–4124. MR 2395165

[Ben19] ———, *Dynamics in one non-archimedean variable*, Graduate Studies in Mathematics, vol. 198, American Mathematical Society, Providence, RI, 2019. MR 3890051

[Ber00] Walter Bergweiler, *The role of the Ahlfors five islands theorem in complex dynamics*, Conform. Geom. Dyn. **4** (2000), 22–34. MR 1741773

[BH88] Bodil Branner and John H. Hubbard, *The iteration of cubic polynomials. I. The global topology of parameter space*, Acta Math. **160** (1988), no. 3-4, 143–206. MR 945011

[Buf03] Xavier Buff, *On the Bieberbach conjecture and holomorphic dynamics*, Proc. Amer. Math. Soc. **131** (2003), no. 3, 755–759. MR 1937413

[Car22] Francisco C. Caramello, Jr., *Introduction to orbifolds*, arXiv:1909.08699v6, 2022.

[CG93] Lennart Carleson and Theodore W. Gamelin, *Complex dynamics*, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993. MR 1230383

[CK97] Jacek Chądzyński and Tadeusz Krasiński, *A set on which the Łojasiewicz exponent at infinity is attained*, Ann. Polon. Math. **67** (1997), no. 2, 191–197. MR 1460600

[DeM16] Laura DeMarco, *Bifurcations, intersections, and heights*, Algebra Number Theory **10** (2016), no. 5, 1031–1056. MR 3531361

[DH84] A. Douady and J. H. Hubbard, *Étude dynamique des polynômes complexes. Partie I*, Publications Mathématiques d'Orsay [Mathematical Publications of Orsay], vol. 84-2, Université de Paris-Sud, Département de Mathématiques, Orsay, 1984. MR 762431

[DM08] Laura G. DeMarco and Curtis T. McMullen, *Trees and the dynamics of polynomials*, Ann. Sci. Éc. Norm. Supér. (4) **41** (2008), no. 3, 337–382. MR 2482442

[Dom97] P. Domínguez, *Connectedness properties of Julia sets of transcendental entire functions*, Complex Variables Theory Appl. **32** (1997), no. 3, 199–215. MR 1457686

[Dom98] ———, *Dynamics of transcendental meromorphic functions*, Ann. Acad. Sci. Fenn. Math. **23** (1998), no. 1, 225–250. MR 1601879

[EL92] A. È. Erëmenko and G. M. Levin, *Estimation of the characteristic exponents of a polynomial*, Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. (1992), no. 58, 30–40. MR 1258059

[Fav24] Charles Favre, *Blow-up of multipliers in meromorphic families of rational maps*, In preparation, 2024.

[FG22] Charles Favre and Thomas Gauthier, *The arithmetic of polynomial dynamical pairs*, Annals of Mathematics Studies, vol. 214, Princeton University Press, Princeton, NJ, 2022. MR 4529887

[FT08] Masayo Fujimura and Masahiko Taniguchi, *A compactification of the moduli space of polynomials*, Proc. Amer. Math. Soc. **136** (2008), no. 10, 3601–3609. MR 2415044

[Fuj07] Masayo Fujimura, *The moduli space of rational maps and surjectivity of multiplier representation*, Comput. Methods Funct. Theory **7** (2007), no. 2, 345–360. MR 2376676

[Gon24] Chen Gong, *Multipliers of rational maps and rescaling limits*, In preparation, 2024.

[Gor16] Igors Gorbovickis, *Algebraic independence of multipliers of periodic orbits in the space of polynomial maps of one variable*, Ergodic Theory Dynam. Systems **36** (2016), no. 4, 1156–1166. MR 3492973

[Got23] Rin Gotou, *Dynamical systems of correspondences on the projective line II: degrees of multiplier maps*, arXiv:2309.15404v1, 2023.

[GOV20] Thomas Gauthier, Yûsuke Okuyama, and Gabriel Vigny, *Approximation of non-archimedean Lyapunov exponents and applications over global fields*, Trans. Amer. Math. Soc. **373** (2020), no. 12, 8963–9011. MR 4177282

[Gro65] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 24, 231. MR 199181

[HT13] Benjamin Hutz and Michael Tepper, *Multiplier spectra and the moduli space of degree 3 morphisms on* $\mathbb{P}^1$, JP J. Algebra Number Theory Appl. **29** (2013), no. 2, 189–206. MR 3136590

[Hub06] John Hamal Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1*, Matrix Editions, Ithaca, NY, 2006, Teichmüller theory, With contributions by Adrien Douady, William Dunbar, Roland Roeder, Sylvain Bonnot, David

Brown, Allen Hatcher, Chris Hruska and Sudeb Mitra, With forewords by William Thurston and Clifford Earle. MR 2245223

[Ing12] Patrick Ingram, *A finiteness result for post-critically finite polynomials*, Int. Math. Res. Not. IMRN (2012), no. 3, 524–543. MR 2885981

[JX23] Zhuchao Ji and Junyi Xie, *Homoclinic orbits, multiplier spectrum and rigidity theorems in complex dynamics*, Forum Math. Pi **11** (2023), Paper No. e11, 37. MR 4585467

[JX24] ———, *The multiplier spectrum morphism is generically injective*, arXiv:2309.15382v2, 2024.

[Kiw15] Jan Kiwi, *Rescaling limits of complex rational maps*, Duke Math. J. **164** (2015), no. 7, 1437–1470. MR 3347319

[Lan83] Serge Lang, *Fundamentals of Diophantine geometry*, Springer-Verlag, New York, 1983. MR 715605

[Lan02] ———, *Algebra*, third ed., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002. MR 1878556

[Liu02] Qing Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002, Translated from the French by Reinie Erné, Oxford Science Publications. MR 1917232

[Luo22] Yusheng Luo, *Trees, length spectra for rational maps via barycentric extensions, and Berkovich spaces*, Duke Math. J. **171** (2022), no. 14, 2943–3001. MR 4491710

[Mat86] Hideyuki Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1986, Translated from the Japanese by M. Reid. MR 879273

[McM87] Curt McMullen, *Families of rational maps and iterative root-finding algorithms*, Ann. of Math. (2) **125** (1987), no. 3, 467–493. MR 890160

[McM88] ———, *Automorphisms of rational maps*, Holomorphic functions and moduli, Vol. I (Berkeley, CA, 1986), Math. Sci. Res. Inst. Publ., vol. 10, Springer, New York, 1988, pp. 31–60. MR 955807

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR 1304906

[Mil92] John Milnor, *Remarks on iterated cubic maps*, Experiment. Math. **1** (1992), no. 1, 5–24. MR 1181083

[Mil93] ———, *Geometry and dynamics of quadratic rational maps*, Experiment. Math. **2** (1993), no. 1, 37–83, With an appendix by the author and Lei Tan. MR 1246482

[Mil06] ———, *On Lattès maps*, Dynamics on the Riemann sphere, Eur. Math. Soc., Zürich, 2006, pp. 9–43. MR 2348953

[Mil17] J. S. Milne, *Algebraic groups*, Cambridge Studies in Advanced Mathematics, vol. 170, Cambridge University Press, Cambridge, 2017, The theory of group schemes of finite type over a field. MR 3729270

[MP94] Patrick Morton and Pratiksha Patel, *The Galois theory of periodic points of polynomial maps*, Proc. London Math. Soc. (3) **68** (1994), no. 2, 225–263. MR 1253503

[MS95] Patrick Morton and Joseph H. Silverman, *Periodic points, multiplicities, and dynamical units*, J. Reine Angew. Math. **461** (1995), 81–122. MR 1324210

[Neu99] Jürgen Neukirch, *Algebraic number theory*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 322, Springer-Verlag, Berlin, 1999, Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder. MR 1697859

[Now00] Krzysztof Jan Nowak, *Some elementary proofs of Puiseux's theorems*, Univ. Iagel. Acta Math. (2000), no. 38, 279–282. MR 1812118

[Oku12] Yûsuke Okuyama, *Repelling periodic points and logarithmic equidistribution in non-archimedean dynamics*, Acta Arith. **152** (2012), no. 3, 267–277. MR 2885787

[Pak19a] F. Pakovich, *On mutually semiconjugate rational functions*, Arnold Math. J. **5** (2019), no. 2-3, 339–354. MR 4031360

[Pak19b] Fedor Pakovich, *Recomposing rational functions*, Int. Math. Res. Not. IMRN (2019), no. 7, 1921–1935. MR 3938311

[Poo17] Bjorn Poonen, *Rational points on varieties*, Graduate Studies in Mathematics, vol. 186, American Mathematical Society, Providence, RI, 2017. MR 3729254

[QY09] WeiYuan Qiu and YongCheng Yin, *Proof of the Branner-Hubbard conjecture on Cantor Julia sets*, Sci. China Ser. A **52** (2009), no. 1, 45–65. MR 2471515

[SGA70] *Schémas en groupes. I: Propriétés générales des schémas en groupes*, Lecture Notes in Mathematics, vol. Vol. 151, Springer-Verlag, Berlin-New York, 1970, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3), Dirigé par M. Demazure et A. Grothendieck. MR 274458

[SGA71] *Revêtements étales et groupe fondamental*, Lecture Notes in Mathematics, vol. Vol. 224, Springer-Verlag, Berlin-New York, 1971, Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1), Dirigé par Alexandre Grothendieck. Augmenté de deux exposés de M. Raynaud. MR 354651

[Sil98] Joseph H. Silverman, *The space of rational maps on* $\boldsymbol{P}^1$, Duke Math. J. **94** (1998), no. 1, 41–77. MR 1635900

[Sil07] ———, *The arithmetic of dynamical systems*, Graduate Texts in Mathematics, vol. 241, Springer, New York, 2007. MR 2316407

[Sil12] ———, *Moduli spaces and arithmetic dynamics*, CRM Monograph Series, vol. 30, American Mathematical Society, Providence, RI, 2012. MR 2884382

[Sug17] Toshi Sugiyama, *The moduli space of polynomial maps and their fixed-point multipliers*, Adv. Math. **322** (2017), 132–185. MR 3720796

[Sug23] ———, *The moduli space of polynomial maps and their fixed-point multipliers: II. Improvement to the algorithm and monic centered polynomials*, Ergodic Theory Dynam. Systems **43** (2023), no. 11, 3777–3795. MR 4651585

[VH92] Franco Vivaldi and Spyros Hatjispyros, *Galois theory of periodic orbits of rational maps*, Nonlinearity **5** (1992), no. 4, 961–978. MR 1174226

Department of Mathematics, University of Toronto, Toronto, ON M5S 2E4, Canada
Email address: `valentin.huguin@utoronto.ca`